Recall that ((i) $\Rightarrow$ (iv) with $m(E) < +\infty$ in the Littlewood's first principle)

Let $m(E) < +\infty$ and $\varepsilon > 0$. Then $\exists\, n \in \mathbb{N}$

$U := \bigcup_{i=1}^{n} {}_{0}\, I_i$, disjoint union of finitely many

open intervals such that $m(E \triangle U) < \varepsilon$.

Ex. If $\mathcal{M} \ni E \subseteq (a,b) \subseteq \mathbb{R}$ then

$I_1, \dots, I_n$ above can be so selected to satisfy the additional property that

$E_i \subseteq (a,b)\ \forall i$.

Cor 1. Let $E \subseteq (a, b) \subseteq \mathbb{R}$, measurable and let $\varphi \in \mathcal{S}(E)$: φ is a simple function vanishing outside E. Then, $\forall \varepsilon > 0$, $\exists$ a step-function $\psi : \mathbb{R} \to \mathbb{R}$ and $A \in \mathcal{M}$ with $m(A) < \varepsilon$ such that $\psi = 0$ on $\mathbb{R} \setminus (a, b)$ and

$$\varphi = \psi \text{ on } \mathbb{R} \setminus A.$$

Show further that $\forall \varepsilon > 0$ $\exists$ a continuous function $g : \mathbb{R} \to \mathbb{R}$ and $A' \in \mathcal{M}$ with $m(A') < \varepsilon$ such that $g = 0$ on $\mathbb{R} \setminus (a, b)$ and

$$\varphi = g \text{ on } \mathbb{R} \setminus A.$$

Proof. Let $\varphi = \sum_{i=1}^{n} \alpha_i \chi_{E_i}$

with each $\alpha_i \in \mathbb{R}$ and

$m \ni E_i \subseteq E$.

Littlewood 1st principle implies that $\exists$

$U_i = \bigcup_{j=1}^{n_i} I_j^{(i)}$ with $n_i \in \mathbb{N}$ and disjoint

open intervals such that $m(E_i \triangle U_i) < \frac{\varepsilon}{n}$.

Since $E_i \subseteq (a,b)$, we may assume that all

the intervals $I_j^{(i)} \subseteq (a,b)$ (use

$I_j^{(i)} \cap (a,b)$ in place of $I_j^{(i)}$ if necessary).

Since $E_i \subseteq (a,b)$, Littlewood

1st principle implies that $\exists$

$U_i = \bigcup_{j=1}^{n_i} I_j^{(i)}$ with $n_i \in \mathbb{N}$ and disjoint ope

Set $\psi := \sum_{i=1}^{n} \alpha_i \chi_{U_i}$. Clearly ψ is a step-function vanishing outside (a,b) and, with $A := \bigcup (E_i \triangle U_i)$,

$$\varphi = \psi \text{ on } \mathbb{R} \setminus A$$

(and $m(A) < \varepsilon$). The reader is requested to construct continuous g with the desired properties.

Corollary 2. Same as Cor 1 but drop the assumption $E \subseteq (a,b)$ while add that $m(E) < +\infty$. Then the same conclusion as Cor 1 with (a,b) replaced by $(-N,N)$ for some $N \in \mathbb{N}$.

Proof. $\forall n \in \mathbb{N}$, let $E_n = E \cap (-n,n)$. Then $\mathcal{M} \ni E_n \uparrow E = \bigcup_{n=1}^{\infty} E_n$, and it follows from the Monotone Convergence Lemma for measures that $\exists N \in \mathbb{N}$ s.t.

$$m(E \setminus E_N) = m(E) - m(E_N) < \varepsilon$$

Let $\varphi_N = \varphi \cdot \chi_{E_N}$ (so φ_N is a simple function vanishing outside $E_N \subseteq (-N,N)$ and $\varphi_N = \varphi$ on $\mathbb{R} \setminus (E \setminus E_N)$). Now apply Cor 1 (to φ_N in place of φ) to obtain the corresponding ψ_N, g_N, A_N & A_N'

Set $\psi := \psi_N$, $g := g_N$, $A := A_N \cup (E \setminus E_N)$, $A' = A'_N \cup (E \setminus E_N)$. Then they have the desired properties but A, A' are of measure $< 2\varepsilon$ (rather than ε). ~~However~~, as $\varepsilon > 0$ was arbitrary, we are done.

Cor 3. Same as in Cor 1 but drop the assumptions

$$m(E) < +\infty, \quad E \subseteq (a,b) \subseteq \mathbb{R},$$

that is, E is only assumed to be of $m(E) \leq +\infty$.
Let $\varepsilon > 0$. Then $\exists\, \psi, g : \mathbb{R} \to \mathbb{R}$ and $A, A' \subseteq \mathbb{R}$ with $m(A), m(A') < \varepsilon$ such that

$$\varphi = \psi \text{ on } \mathbb{R} \setminus A$$
$$\varphi = g \text{ on } \mathbb{R} \setminus A',$$

and that g is continuous and the restriction of ψ to any finite interval is a step-function.

Proof. $\forall n \in \mathbb{N}$, let $E_n := E \cap (n-1, n) \in \mathcal{M}$ and note that $E \subseteq \bigcup_{n \in \mathbb{Z}} E_n \cup \mathbb{Z}$. Note that $m(\mathbb{Z}) = 0$ and

$$\sum_{n \in \mathbb{Z}} \frac{1}{2^{|n|}} = \sum_{n \in \mathbb{N}} \frac{1}{2^{|-n|}} + \frac{1}{2^0} + \sum_{n \in \mathbb{N}} \frac{1}{2^n} = 3.$$

By Cor 1, $\forall \varepsilon > 0$, $\exists$ step-function ψ_n and continuous function g_n vanishing outside $(n-1, n)$, and A_n, A'_n of measures $< \dfrac{\varepsilon}{4 \cdot 2^{|n|}}$

such that
$$\varphi = \psi_n \text{ on } \mathbb{R} \setminus (n-1, n),$$
$$\varphi = g_n \text{ on } \mathbb{R} \setminus (n-1, n).$$

Set $\psi = \sum_{n \in \mathbb{Z}} \psi_n$, that is, $\forall x \in \mathbb{R}$,
$$\psi(x) = \sum_{n \in \mathbb{Z}} \psi_n(x) \quad (\text{only one term possibly nonzero});$$
note in particular $\psi = 0$ on $\mathbb{Z}$. Similarly one defines $g := \sum_{n \in \mathbb{Z}} g_n$. Then, with
$$A = \bigcup_{n \in \mathbb{Z}} A_n \left(\text{of mea} < \frac{3\varepsilon}{4} < \varepsilon\right)$$
$$A' = \bigcup_{n \in \mathbb{Z}} A'_n \left(- - - - - \right),$$
they have the desired properties.

Cor 1. Let $E \subseteq (a,b) \subseteq \mathbb{R}$, measurable and let $\varphi \in \mathcal{S}(E)$:
φ is a simple function vanishing outside E. Then, $\forall \varepsilon > 0$, $\exists$ a step-function $\psi : \mathbb{R} \to \mathbb{R}$ and $A \in \mathcal{M}$ with $m(A) < \varepsilon$ such that $\psi = 0$ on $\mathbb{R} \setminus (a,b)$ and

$$\varphi = \psi \text{ on } E \setminus A.$$

proof. Let $\varphi := \sum_{i=1}^{n} \alpha_i \chi_{E_i}$

with each $\alpha_i \in \mathbb{R}$ and $\mathcal{M} \ni E_i \subseteq E$

Since $E_i \subseteq (a,b)$, Littlewood's 1st principle implies that $\exists$

$U_i = \bigcup_{j=1}^{n_i} I_j^{(i)}$ with $n_i \in \mathbb{N}$ and disjoint open intervals $I_1^{(i)}, \dots, I_{n_i}^{(i)}$ contained in (a,b) (replace these intervals by their intersections with (a,b) if necessary) such that $m(E_i \triangle U_i) < \frac{\varepsilon}{n}$, so χ_{U_i} is a step-function vanishing on $\mathbb{R} \setminus (a,b)$ and $\chi_{E_i} = \chi_{U_i}$ on $\mathbb{R} \setminus (E_i \triangle U_i)$.

Now let $A := \bigcup_{i=1}^{n} (E_i \triangle U_i)$, $\psi := \sum_{i=1}^{n} \alpha_i \chi_{U_i}$; clearly ψ is a step-function vanishing on $\mathbb{R} \setminus (a,b)$ and $m(A) < \varepsilon$ and

$$\varphi = \psi \text{ on } \mathbb{R} \setminus (a,b)$$