MATH2050B Mathematical Analysis I 18/19

Assignment 5

Let (x_n) be a bounded sequence, $y_n := \sup\{\, x_n,\, x_{n+1},\, ...\}$. $v \in \mathbb{R}$ is said to be an **essential upper bound** of (x_n) if $\exists N \in \mathbb{N}$ such that $x_m \leq v \ \forall n \geq N$.
Let

$$V := \{\, v \in \mathbb{R} : v \text{ is an essential upper bound of } (x_n) \,\}$$

and

$$L := \{\, l \in \mathbb{R} : \exists \text{ a subseqeunce of } (x_n) \text{ convergent to } l \,\}$$

Show that

1. By what theorem (how it is stated), you can conclude that $y^* := \lim\limits_n (y_n)$ does exist and $y^* = \inf\{\, y_n : n \in \mathbb{N} \,\}$?

2. Let $\alpha \in \mathbb{R}$. Then
   $$y^* \leq \alpha \text{ iff } \forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } x_n < \alpha + \epsilon \ \ \forall n \geq N$$
   and
   $$\alpha \leq y^* \text{ iff } \forall \epsilon > 0, \forall N \in \mathbb{N}, \exists n > N \text{ s.t. } \alpha - \epsilon < x_n$$

3. Whate are y_n, y^*, V and L if $x_n = \frac{1}{n} \ \forall n$ (do the same for $x_n = 1 - \frac{1}{n} \ \forall n$).

4. Show that any upper bound of (x_n) is an essential upper bound of (x_n), and that any lower bound of (x_n) is a lower bound of V so $\inf V$ exists in $\mathbb{R}$.

5. $\inf V = max\ L = y^*$ (denoted by $\lim\limits_n \sup x_n$)