# TA's solution to 2050B homework 5

1. (2 marks) Monotone convergence theorem states that if $(\xi_n)$ is a bounded decreasing sequence, then $\lim \xi_n$ exists and $\lim \xi_n = \inf\{\xi_n : n \in \mathbb{N}\}$. Apply it to $(y_n)$, the result follows.

2. (4 marks)

   (a) $(\Rightarrow)$
   Suppose $y^* \leq \alpha$. Let $\varepsilon > 0$. Since $\lim y_n = y^*$, $\exists N \in \mathbb{N}$ such that $y_n < \alpha + \varepsilon$ for all $n \geq N$ (c.f. homework II 4(b)). In particular, $\sup\{x_N, x_{N+1}, \ldots\} = y_N < \alpha + \varepsilon$. Therefore $\alpha + \varepsilon$ is greater than an upper bound of $\{x_N, x_{N+1}, \ldots\}$. It follows that $x_n < \alpha + \varepsilon$ for all $n \geq N$.

   $(\Leftarrow)$
   Let $\varepsilon > 0$. By assumption, $\exists N \in \mathbb{N}$ such that $x_n < \alpha + \varepsilon$ for all $n \geq N$. Therefore $\alpha + \varepsilon$ is an upper bound of the $\{x_N, x_{N+1}, \ldots\}$, so $\alpha + \varepsilon \geq \sup\{x_N, x_{N+1}, \ldots\} = y_N$. Since $(y_n)$ is a decreasing sequence, we have $\alpha + \varepsilon \geq y_n$ whenever $n \geq N$. Hence

   $$\alpha + \varepsilon = \lim_{n\to\infty} (\alpha + \varepsilon) \geq \lim_{n\to\infty} (y_n) = y^*.$$

   Since $\varepsilon > 0$ is arbitrary, we conclude that $\alpha \geq y^*$.

   (b) $(\Rightarrow)$
   Suppose $\alpha \leq y^*$. Let $\varepsilon > 0$. Since $\lim y_n = y^*$, $\exists M \in \mathbb{N}$ such that $\alpha - \varepsilon < y_n$ for all $n \geq M$ (c.f. homework II 4(b)). Now for any $N \in \mathbb{N}$, we have $\alpha - \varepsilon < y_{\max(M,N+1)} = \sup\{x_{\max(M,N+1)}, x_{\max(M,N+1)+1}, \ldots\}$. As $\alpha - \varepsilon$ is less than the supremum of $\{x_{\max(M,N+1)}, x_{\max(M,N+1)+1}, \ldots\}$, it fails to be an upper bound of that set. Hence, $\exists n \geq \max(M, N+1)$ such that $\alpha - \varepsilon < x_n$. Done.

   $(\Leftarrow)$
   Let $\varepsilon > 0$. Given $N \in \mathbb{N}$, by assumption $\exists \ell > N$ such that $\alpha - \varepsilon < x_\ell$. Then $\alpha - \varepsilon \leq \sup\{x_N, x_{N+1}, \ldots, x_\ell, x_{\ell+1}, \ldots\} = y_N$. Since this holds for all $N \in \mathbb{N}$, we have

$$\alpha - \varepsilon = \lim_{n\to\infty}(\alpha - \varepsilon) \leq \lim_{n\to\infty}(y_n) = y^*.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\alpha \leq y^*$.

3. (2 marks)

   (a) When $(x_n) = (1/n)$,

   - $y_n = 1/n$
   - $y^* = 0$
   - $V = (0, \infty)$
   - $L = \{0\}$

   (b) When $(x_n) = (1 - 1/n)$,

   - $y_n = 1$
   - $y^* = 1$
   - $V = [1, \infty)$
   - $L = \{1\}$

4. (2 marks)

   Let $u$ be an upper bound of $(x_n)$. Then $x_n \leq u\ \forall n \geq 1$. Referring to the definition, we see that $u$ is an essential upper bound of $(x_n)$.

   Let $l$ be a lower bound of $(x_n)$. Then $l \leq x_n\ \forall n \geq 1$. If $v \in V$, then $\exists N$ s.t. $x_n \leq v\ \forall n \geq N$. Now $l \leq x_N \leq v$. Since $v \in V$ is chosen arbitrarily, this shows that $l$ is a lower bound of $V$.

   Since $(x_n)$ is bounded, it has an upper and lower bound. Therefore, by the first paragraph, $V$ is non-empty, while by the second paragraph, $V$ is bounded below. Hence, $\inf V$ exists in $\mathbb{R}$.

5. We show the following:

   - $\inf V = y^*$;
   - There exists a subsequence $(x_{n_k})$ of $(x_n)$ such that $\lim_k x_{n_k} = y^*$;
   - $\sup L = y^*$.

(Since $y^* \in L$ by the second statement, it then follows from the third statement that $\max L = y^*$)

(a) To show $\inf V = y^*$, we go to show that $y^*$ is a lower bound of $V$, while $y^* + \varepsilon$ fails to be a lower bound of $V$ for any $\varepsilon > 0$.

   Let $v \in V$. By the definition of $V$, there exists $N_v \in \mathbb{N}$ such that $x_n \leq v \ \forall n \geq N_v$. It follows from question 2(a) that $y^* \leq v$. Since $v \in V$ is arbitrarily chosen, we see that $y^*$ is a lower bound of $V$.

   On the other hand, given $\varepsilon > 0$, since $\lim y_n = y^*$, there exists $N \in \mathbb{N}$ such that $y_N \leq y^* + \varepsilon/2$ (c.f. homework II question 4(b)). Therefore, by the definition of $y_N$, we have $x_n \leq y^* + \varepsilon/2$ for all $n \geq N$. This means $y^* + \varepsilon/2$ is an essential upper bound of $(x_n)$, i.e. $y^* + \varepsilon/2 \in V$. Hence $y^* + \varepsilon$ cannot be a lower bound of $V$.

(b) By question 2(b), $\exists n_1 > 1$ such that $x_{n_1} > y^* - 1$ (take $\alpha = y^*$, $\varepsilon = 1$, $N = 1$).
   Next, by question 2(b), $\exists n_2 > n_1$ such that $x_{n_2} > y^* - 1/2$ (take $\alpha = y^*$, $\varepsilon = 1/2$, $N = n_1$).
   Then, by question 2(b) again, $\exists n_3 > n_2$ such that $x_{n_3} > y^* - 1/3$ (take $\alpha = y^*$, $\varepsilon = 1/3$, $N = n_2$).
   Continuing this process by induction, we construct a subsequence $(x_{n_k})$ of $(x_n)$ such that $x_{n_k} \geq y^* - 1/k$ for all $k$. Since

$$y^* - \frac{1}{k} \leq x_{n_k} \leq \sup\{x_{n_k}, x_{n_k+1}, \ldots\} = y_{n_k},$$

   and that

$$\lim_{k\to\infty} (y^* - \frac{1}{k}) = y^* = \lim_{k\to\infty} y_{n_k},$$

   we conclude that $(x_{n_k})$ converges to $y^*$ by squeeze theorem.

(c) To show $\sup L = y^*$, we go to show that $y^*$ is an upper bound of $L$, while $y^* - \varepsilon$ fails to be an upper bound of $L$ for any $\varepsilon > 0$.

   Let $\ell \in L$. Then by the definition of $L$, there exists $(x_{n_k})$ such that $\lim_k x_{n_k} = \ell$. Since $x_{n_k} \leq \sup\{x_{n_k}, x_{n_k+1}, \ldots\} = y_{n_k}$, it follows that

$$\ell = \lim_{k\to\infty} x_{n_k} \leq \lim_{k\to\infty} y_{n_k} = y^*.$$

Since $\ell \in L$ is arbitrarily chosen, we see that $y^*$ is an upper bound of $L$.

On the other hand, by (b), we see that $y^* \in L$, so $y^* - \varepsilon$ cannot be an upper bound of $L$ for any $\varepsilon > 0$.