According to definition (2) of $f(z)$,

$$\left|\int_{C_\rho} f(z)\,dz\right| \le \frac{\rho^{-a}}{1-\rho}2\pi\rho = \frac{2\pi}{1-\rho}\rho^{1-a}$$

and

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{R^{-a}}{R-1}2\pi R = \frac{2\pi R}{R-1}\cdot\frac{1}{R^a}.$$

Since $0 < a < 1$, the values of these two integrals evidently tend to 0 as ρ and R tend to 0 and ∞, respectively. Hence, if we let ρ tend to 0 and then R tend to ∞ in equation (4), we arrive at the result

$$(1 - e^{-i2a\pi})\int_0^\infty \frac{r^{-a}}{r+1}\,dr = 2\pi i e^{-ia\pi},$$

or

$$\int_0^\infty \frac{r^{-a}}{r+1}\,dr = 2\pi i \frac{e^{-ia\pi}}{1-e^{-i2a\pi}}\cdot\frac{e^{ia\pi}}{e^{ia\pi}} = \pi\frac{2i}{e^{ia\pi}-e^{-ia\pi}}.$$

Using the variable of integration x here, instead of r, as well as the expression

$$\sin a\pi = \frac{e^{ia\pi}-e^{-ia\pi}}{2i},$$

we arrive at the desired result:

(5) $$\int_0^\infty \frac{x^{-a}}{x+1}\,dx = \frac{\pi}{\sin a\pi} \qquad (0 < a < 1).$$

EXERCISES

1. Use the function $f(z) = (e^{iaz} - e^{ibz})/z^2$ and the indented contour in Fig. 108 (Sec. 89) to derive the integration formula

$$\int_0^\infty \frac{\cos(ax)-\cos(bx)}{x^2}\,dx = \frac{\pi}{2}(b-a) \qquad (a \ge 0, b \ge 0).$$

Then, with the aid of the trigonometric identity $1 - \cos(2x) = 2\sin^2 x$, point out how it follows that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

2. Derive the integration formula

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$$

by integrating the function

$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{e^{(-1/2)\log z}}{z^2+1} \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

over the indented contour appearing in Fig. 109 (Sec. 90).

3. Derive the integration formula obtained in Exercise 2 by integrating the branch

$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{e^{(-1/2)\log z}}{z^2+1} \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

of the multiple-valued function $z^{-1/2}/(z^2+1)$ over the closed contour in Fig. 110 (Sec. 91).

4. Derive the integration formula

$$\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}\,dx = \frac{2\pi}{\sqrt{3}}\cdot\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b} \qquad (a > b > 0)$$

using the function

$$f(z) = \frac{z^{1/3}}{(z+a)(z+b)} = \frac{e^{(1/3)\log z}}{(z+a)(z+b)} \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

and a closed contour similar to the one in Fig. 110 (Sec. 91), but where

$$\rho < b < a < R.$$

5. The ***beta function*** is this function of two real variables:

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt \qquad (p > 0, q > 0).$$

Make the substitution $t = 1/(x+1)$ and use the result obtained in the example in Sec. 91 to show that

$$B(p, 1-p) = \frac{\pi}{\sin(p\pi)} \qquad (0 < p < 1).$$

6. Consider the two simple closed contours shown in Fig. 111 and obtained by dividing into two pieces the annulus formed by the circles C_ρ and C_R in Fig. 110 (Sec. 91). The legs L and $-L$ of those contours are directed line segments along any ray $\arg z = \theta_0$, where $\pi < \theta_0 < 3\pi/2$. Also, Γ_ρ and γ_ρ are the indicated portions of C_ρ, while Γ_R and γ_R make up C_R.

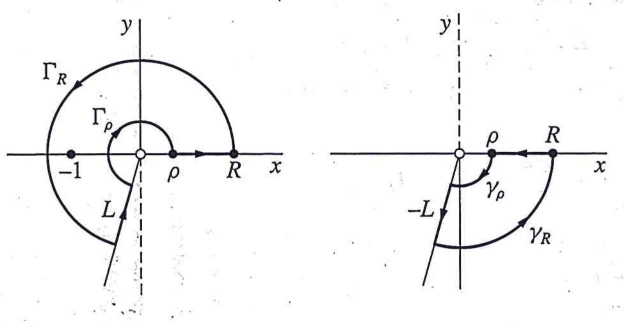

FIGURE 111

(a) Show how it follows from Cauchy's residue theorem that when the branch

$$f_1(z) = \frac{z^{-a}}{z+1} \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

where C_R is the closed semicircular path shown in Fig. 105. By equating imaginary parts on each side of equation (4), we arrive at

$$(5)\qquad \int_{-R}^{R} \frac{x \sin 2x}{x^2+3}\,dx = \pi \exp(-2\sqrt{3}) - \operatorname{Im} \int_{C_R} f(z)e^{i2z}\,dz,$$

Now the property $|\operatorname{Im} z| \le |z|$ of complex numbers tells us that

$$(6)\qquad \left|\operatorname{Im} \int_{C_R} f(z)e^{i2z}\,dz\right| \le \left|\int_{C_R} f(z)e^{i2z}\,dz\right|;$$

and we note that when z is a point on C_R,

$$|f(z)| \le M_R \quad \text{where} \quad M_R = \frac{R}{R^2-3}$$

and that $|e^{i2z}| = e^{-2y} \le 1$ for such a point.

By proceeding as we did in Sec. 87, we *cannot* conclude that the right-hand side of inequality (6) tends to 0 as R tends to ∞. This is because the quantity

$$M_R \pi R = \frac{\pi R^2}{R^2-3} = \frac{\pi}{1-\dfrac{3}{R^2}}$$

does not tend to zero.

The theorem at the beginning of this section does, however, provide the desired limit:

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{i2z}dz = 0.$$

This is because

$$M_R = \frac{\dfrac{1}{R}}{1-\dfrac{3}{R^2}} \to 0 \quad \text{as} \quad R \to \infty.$$

So it does, indeed, follow from inequality (6) that the left-hand side there tends to zero as R tends to infinity. Consequently, since the integrand on the left in equation (5) is even, we arrive at the result

$$\int_{-\infty}^{\infty} \frac{x \sin 2x}{x^2+3}\,dx = \pi \exp(-2\sqrt{3}),$$

or

$$\int_{0}^{\infty} \frac{x \sin 2x}{x^2+3}\,dx = \frac{\pi}{2} \exp(-2\sqrt{3}).$$

EXERCISES

Use residues to derive the integration formulas in Exercises 1 through 5.

1. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right) \quad (a > b > 0).$

2. $\displaystyle\int_{0}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2}e^{-a} \quad (a > 0).$

3. $\displaystyle\int_{0}^{\infty} \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{\pi}{4b^3}(1+ab)e^{-ab} \quad (a > 0, b > 0).$

4. $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin ax}{x^4+4}\,dx = \frac{\pi}{2}e^{-a}\sin a \quad (a > 0).$

5. $\displaystyle\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4+4}\,dx = \pi e^{-a}\cos a \quad (a > 0).$

Use residues to evaluate the integrals in Exercises 6 and 7.

6. $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x\,dx}{(x^2+1)(x^2+4)}.$

7. $\displaystyle\int_{0}^{\infty} \frac{x^3 \sin x\,dx}{(x^2+1)(x^2+9)}.$

Use residues to find the Cauchy principal values of the improper integrals in Exercises 8 through 11.

8. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x^2+4x+5}.$

   Ans. $-\dfrac{\pi}{e}\sin 2.$

9. $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x\,dx}{x^2+2x+2}.$

   Ans. $\dfrac{\pi}{e}(\sin 1 + \cos 1).$

10. $\displaystyle\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5}\,dx.$

    Ans. $\dfrac{\pi}{e}(\sin 2 - \cos 2).$

11. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x+a)^2+b^2} \quad (b > 0).$

12. Follow the steps below to evaluate the ***Fresnel integrals***, which are important in diffraction theory:

$$\int_{0}^{\infty} \cos(x^2)\,dx = \int_{0}^{\infty} \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

Next, we show that the integral on the right in equation (3) tends to 0 as R tends to ∞. To do this, we observe that when $R > 1$,

$$|z^6 + 1| \ge \big| |z|^6 - 1 \big| = R^6 - 1.$$

So, if z is any point on C_R,

$$|f(z)| = \frac{1}{|z^6+1|} \le M_R \quad \text{where} \quad M_R = \frac{1}{R^6 - 1};$$

and this means that

$$\left| \int_{C_R} f(z)\,dz \right| \le M_R\, \pi R, \tag{4}$$

πR being the length of the semicircle C_R. (See Sec. 47.) Since the number

$$M_R\, \pi R = \frac{\pi R}{R^6 - 1}$$

is a quotient of polynomials in R and since the degree of the numerator is less than the degree of the denominator, that quotient must tend to zero as R tends to ∞. More precisely, if we divide both numerator and denominator by R^6 and write

$$M_R \pi R = \frac{\dfrac{\pi}{R^5}}{1 - \dfrac{1}{R^6}},$$

it is evident that $M_R \pi R$ tends to zero. Consequently, in view of inequality (4),

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0.$$

It now follows from equation (3) that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^6+1} = \frac{2\pi}{3},$$

or

$$\text{P.V.} \int_{-R}^{R} \frac{dx}{x^6+1} = \frac{2\pi}{3}.$$

Since the integrand here is even, we know from equation (7) in Sec. 85 that

$$\int_0^\infty \frac{dx}{x^6+1} = \frac{\pi}{3}. \tag{5}$$

EXERCISES

Use residues to derive the integration formulas in Exercises 1 through 6.

1. $\displaystyle\int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}.$

2. $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}.$

3. $\displaystyle\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$

4. $\displaystyle\int_0^\infty \frac{x^2\,dx}{x^6+1} = \frac{\pi}{6}.$

5. $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}.$

6. $\displaystyle\int_0^\infty \frac{x^2\,dx}{(x^2+9)(x^2+4)^2} = \frac{\pi}{200}.$

Use residues to find the Cauchy principal values of the integrals in Exercises 7 and 8.

7. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2+2x+2}.$

8. $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{(x^2+1)(x^2+2x+2)}.$

   Ans. $-\pi/5$.

9. Use a residue and the contour shown in Fig. 101, where $R > 1$, to establish the integration formula

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi}{3\sqrt{3}}.$$

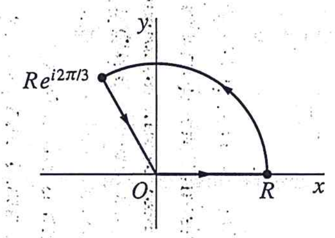

FIGURE 101

10. Let m and n be integers, where $0 \le m < n$. Follow the steps below to derive the integration formula

$$\int_0^\infty \frac{x^{2m}}{x^{2n}+1}\,dx = \frac{\pi}{2n} \csc\left(\frac{2m+1}{2n}\pi\right).$$

   (*a*) Show that the zeros of the polynomial $z^{2n} + 1$ lying above the real axis are

$$c_k = \exp\left[i\frac{(2k+1)\pi}{2n}\right] \qquad (k = 0, 1, 2, \ldots, n-1)$$

   and that there are none on that axis.

   (*b*) With the aid of Theorem 2 in Sec. 83, show that

$$\operatorname*{Res}_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = -\frac{1}{2n} e^{i(2k+1)\alpha} \qquad (k = 0, 1, 2, \ldots, n-1)$$

Note that because $|a| < 1$,

$$|z_2| = \frac{1+\sqrt{1-a^2}}{|a|} > 1.$$

Also, since $|z_1 z_2| = 1$, it follows that $|z_1| < 1$. Hence there are no singular points on C, and the only one interior to it is the point z_1. The corresponding residue B_1 is found by writing

$$f(z) = \frac{\phi(z)}{z - z_1} \quad \text{where} \quad \phi(z) = \frac{2/a}{z - z_2}.$$

This shows that z_1 is a simple pole and that

$$B_1 = \phi(z_1) = \frac{2/a}{z_1 - z_2} = \frac{1}{i\sqrt{1-a^2}}.$$

Consequently,

$$\int_C \frac{2/a}{z^2 + (2i/a)z - 1}\, dz = 2\pi i B_1 = \frac{2\pi}{\sqrt{1-a^2}};$$

and integration formula (5) follows.

The method just illustrated applies equally well when the arguments of the sine and cosine are integral multiples of θ. One can use equation (2) to write, for example,

(7) $$\cos 2\theta = \frac{e^{i2\theta} + e^{-i2\theta}}{2} = \frac{(e^{i\theta})^2 + (e^{i\theta})^{-2}}{2} = \frac{z^2 + z^{-2}}{2}.$$

EXAMPLE 2. Our goal here is to show that

(8) $$\int_0^\pi \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2} = \frac{a^2\pi}{1-a^2} \qquad (-1 < a < 1).$$

Just as we did in Example 1, we exclude the possibility that $a = 0$, in which case equation (8) is obviously true. We begin with the observation that because

$$\cos(2\pi - \theta) = \cos\theta \quad \text{and} \quad \cos 2(2\pi - \theta) = \cos 2\theta,$$

the graph of the integrand is symmetric with respect to the vertical line $\theta = \pi$. This observation, together with equations (3) and (7), enables us to write

$$\int_0^\pi \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2} = \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2} = \frac{i}{4}\int_C \frac{z^4+1}{(z-a)(az-1)z^2}\, dz,$$

where C is the positively oriented circle in Fig. 112. Evidently, then,

(9) $$\int_0^\pi \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2} = \frac{i}{4} 2\pi i (B_1 + B_2),$$

where B_1 and B_2 denote the residues of the function

$$f(z) = \frac{z^4+1}{(z-a)(az-1)z^2}$$

at a and 0, respectively. The singularity $z = 1/a$ is, of course, exterior to the circle C since $|a| < 1$.

Inasmuch as

$$f(z) = \frac{\phi(z)}{z-a} \quad \text{where} \quad \phi(z) = \frac{z^4+1}{(az-1)z^2},$$

it is easy to see that

(10) $$B_1 = \phi(a) = \frac{a^4+1}{(a^2-1)a^2}.$$

The residue B_2 can be found by writing

$$f(z) = \frac{\phi(z)}{z^2} \quad \text{where} \quad \phi(z) = \frac{z^4+1}{(z-a)(az-1)};$$

and straightforward differentiation reveals that

(11) $$B_2 = \phi'(0) = \frac{a^2+1}{a^2}.$$

Finally, by substituting the residues (10) and (11) into expression (9), we arrive at the integration formula (8).

EXERCISES

Use residues to establish the following integration formulas:

1. $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} = \frac{2\pi}{3}.$

2. $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2\theta} = \sqrt{2}\pi.$

3. $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta\, d\theta}{5 - 4\cos 2\theta} = \frac{3\pi}{8}.$

4. $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}} \qquad (-1 < a < 1).$

5. $\displaystyle\int_0^{\pi} \frac{d\theta}{(a + \cos\theta)^2} = \frac{a\pi}{(\sqrt{a^2-1})^3} \qquad (a > 1).$

6. $\displaystyle\int_0^{\pi} \sin^{2n}\theta\, d\theta = \frac{(2n)!}{2^{2n}(n!)^2}\pi \qquad (n = 1, 2, \ldots).$

93. ARGUMENT PRINCIPLE

A function f is said to be ***meromorphic*** in a domain D if it is analytic throughout D except for poles. Suppose now that f is meromorphic in the domain interior to a positively oriented simple closed contour C and that it is analytic and nonzero on C.