THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
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Tutorial 2

Definition 1. *The derivative of f at z_0 is the limit*

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \quad \textit{(h is non-zero complex number)}$$

and f is said to be differentiable at z_0 when this limit exists. The derivative is denoted by $f'(z)$ or $\frac{d}{dz}f(z)$.

Proposition 1. *(Property of differentiation)*

- $\frac{d}{dz}z^n = nz^{n-1}$ *if n is a integer ($z \neq 0$ if the integer is negative)*
- $\frac{d}{dz}[cf(z)] = cf'(z)$ *if c is a constant*
- $\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$
- $\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$
- *Let $F(z) = g(f(z))$ and $w = f(z)$, then $F'(z) = g'(w)f'(z)$.*

Theorem 1. *Suppose that $f(z) = u(x, y) + iv(x, y)$ and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v exist at (x_0, y_0) and they satisfy the Cauchy-Riemann equations*

$$u_x = v_y \ , \ u_y = -v_x.$$

And we can write $f'(z_0) = u_x + iv_x$.

Proof. (Sketch) By the definition of derivative,

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

where h is a non-zero complex number. By definition of limit, the limiting values are the same if h approaches 0 by any paths. If we take $h = r$ and $h = ir$ for non-zero real number r respectively, then by comparing the real and imaginary parts of the limits we obtain the results. □

The converse is also true:

Theorem 2. *Suppose that $f(z) = u(x, y) + iv(x, y)$. Suppose u and v have continuous first-order partial derivatives at (x_0, y_0). If u and v satisfy the Cauchy-Riemann equations*

$$u_x = v_y \ , \ u_y = -v_x,$$

then f is differentiable at (x_0, y_0).

Remark : The Cauchy-Riemann equations in polar coordinate is given by $ru_r = v_\theta$ and $u_\theta = -rv_r$. And we have $f'(z) = e^{-i\theta}(u_r + iv_r)$.

Remark : The continuity of the first-order partial derivatives is required here.

Theorem 3. *Suppose f is defined in a connected open set U. f is analytic in U (holomorphic or it can be differentiated infinitely many times in complex sense) if and only if it is differentiable in U.*

Remark : If a function is differentiable at a point, then it is not analytic at that point. Analyticity must be defined in a open set (a neighborhood of a point).

Definition 2. *An entire function is a function that is analytic at each point in the entire complex plane.*

$f = u + iv$ is said to be real differentiable if u and v are real differentiable with respect to x and y. From Cauchy-Riemann equations we know that complex differentiability implies real differentiability. But the converse is not true, complex differentiability is stronger than real differentiability. The condition for real differentiability does not give us the Cauchy-Riemann equations.

Example 1.

$$f(z) = |z|^2 = u + iv = x^2 + y^2$$

It gives us $u_x = 2x$ and $v_y = 0$.

Example 2.

$$f(z) = u + iv = \overline{z} = x - iy$$

Theorem 4. *(Schwarz Reflection Principle) Let G be a open connected set such that $G = \{z \in \mathbb{C} | \overline{z} \in G\}$ (the set G is symmetric with respect to real axis). Denote $G_+ = G \cap \{Im(z) > 0\}$, $G_- = G \cap \{Im(z) < 0\}$ and $G_0 = G \cap \{Im(z) = 0\}$. Assume f is analytic in G, then $f(z) = \overline{f(\overline{z})}$ if and only if f is real on G_0.*

Remark : One of the sides of the theorem is trivial. The importance of this theorem tells us that we can obtain the value of the function in G from only $G_+ \cup G_0$(or $G_- \cup G_0$) .

Exercise :

1. Prove the Cauchy-Riemann equations in polar form.

2. Determine where $f'(z)$ exists and find its value (a) $f = z^2 + iy^2$,(b)$f = zlm(z)$.

3. Given $f = u + iv$ and $u = xy$, find v such that f is analytic.

4. Let G be a open connected set such that $G = \{z \in \mathbb{C} | \overline{z} \in G\}$. Denote $G_+ = G \cap \{Im(z) > 0\}$,$G_- = G \cap \{Im(z) < 0\}$ and $G_0 = G \cap \{Im(z) = 0\}$. Assume f is analytic in G, then $f(z) = -\overline{f(\overline{z})}$ if and only if f is pure imaginary on G_0.