THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018 Spring MATH2230
Tutorial 7

0.1 Some applications of Cauchy Integral Formula

Theorem 1. *(Cauchy's estimate) Suppose that a function f is analytic inside and on a positively oriented circle $C_R = \{z \in \mathbb{C} \mid |z - z_0| = R\}$. If M_R denotes the maximum value of $|f(z)|$ on C_R, then*

$$|f^{(n)}(z_0)| \leq \frac{n! M_R}{R^n}$$

Remark : It is a immediate consequence of generalized cauchy integral formula.
Remark : The maximum value M_R must exist since C_R is compact.

Theorem 2. *(Liouville's theorem) If f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.*

Remark : The proof is easy using Cauchy's estimate. If f is bounded, then the constant $M_R = M$ is independent of R. We have $|f'(z_0)| \leq \dfrac{M}{R}$ for any z_0 and $R > 0$, by taking $R \to \infty$, we have $f'(z_0) = 0$. Hence f is constant.

Remark : An important consequence is that entire function can not be bounded !(compare to real variable function) Since entire must be bounded on compact set, so entire function becomes infinite at infinite. (Unless it is a constant function)

Theorem 3. *(Fundamental Theorem of Algebra) If $p(z)$ is non-constant polynomial, then there is a complex number a with $p(a) = 0$*

Proof. We prove by contradiction. Suppose there is no $a \in \mathbb{C}$ such that $p(a) = 0$. Thus $p(z) \neq 0$ in $\mathbb{C}$, then $f = p^{-1}$ is entire. Suppose

$$p = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n = z^n (a_0 z^{-n} + a_1 z^{-(n-1)} + ... + a_n)$$

Thus $\lim_{z\to\infty} p = \infty$ which implies $\lim_{z\to\infty} f = 0$. Since f is entire, then it must be continuous. We can find a large $R > 0$ such that $|f(z)| < 1$ if $|z| > R$. Since f is continuous on $\overline{B_R(0)}$, then it is bounded in $\overline{B_R(0)}$, says, $|f(z)| < M$ if $|z| \leq R$. Hence f is bounded thereofre by Liouville's theorem, $f = p^{-1}$ is constant, which contradicts to our assumption. □

Remark : It is a very short proof of Fundamental Theorem of Algebra by using complex analysis. The proof will be very long and hard if we use algebric method. (MATH3040 will introduce this proof)

Theorem 4. *(Maximum Modulus principle) Suppose that $|f(z)| \leq |f(z_0)|$ at each point $z \in B_\varepsilon(z_0)$ in which f is analytic. Then $f(z) = f(z_0)$ is constant throughout $B_\varepsilon(z_0)$.*

Proof. We let $0 < r < \varepsilon$ and set $z = z_0 + re^{i\theta}$, we have by Cauchy Integral Formula

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(z)dz}{z - z_0} = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$$

By assumption, $|f(z_0 + re^{i\theta})| \leq |f(z_0)|$ We have

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta \leq |f(z_0)|$$

So it must hold that

$$|f(z_0)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta$$

and hence

$$\frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| - |f(z_0 + re^{i\theta})| d\theta = 0$$

Since $|f(z_0)| - |f(z_0 + re^{i\theta})| \geq 0$ by assumption, then we must have $|f(z_0)| = |f(z_0 + re^{i\theta})|$ for any $0 < r < \varepsilon$. Hence $C = |f(z_0)| = |f(z)|$ for any $z \in B_\varepsilon(z_0)$. So $f(z)$ must be constant. (If $|f(z)|$ is constant in U, then $f(z)$ is constant in U.) □

Remark : The theorem is true that if f is analytic and $|f(z)| \leq |f(z_0)|$ at each point in a open connected domain.

Remark : It is equivalent to say that if f is non-constant analytic function and $|f(z)| \leq |f(z_0)|$ at each point in a open connected domain, then there is no point z_0 in the domain such that $|f(z)| \leq |f(z_0)|$ for all z in the domain.

Remark : Under the assumption of this theorem, we can say the maximum value must appear at the boundary of the domain.

0.2 Exercise:

1. If f is analytic in open set U and $|f(z)|$ is constant in U, then $f(z)$ is constant in U.
2. Let $f = \sum_0^\infty a_n z^n$ be entire such that $|f(z)| \leq A|z|$ for all z, where A is fixed constant. Show that $f = az$ where a is a constant. (Hint: Consider derivatives of f)
3. Let $f = u + iv$ be entire and $u \leq M$ in $\mathbb{C}$, then u must be constant. (Hint: Consider e^f)
4. Let f be non-constant analytic in open connected U. Suppose $f \neq 0$ in $\overline{U}$, prove that $|f|$ can not attain its minimum value in U.