6. Show that the Bessel equation of order one-half

$$x^2y'' + xy' + \left(x^2 - \tfrac{1}{4}\right)y = 0, \qquad x > 0$$

can be reduced to the equation

$$v'' + v = 0$$

by the change of dependent variable $y = x^{-1/2}v(x)$. From this, conclude that $y_1(x) = x^{-1/2}\cos x$ and $y_2(x) = x^{-1/2}\sin x$ are solutions of the Bessel equation of order one-half.

7. Show directly that the series for $J_0(x)$, Eq. (7), converges absolutely for all x.

8. Show directly that the series for $J_1(x)$, Eq. (27), converges absolutely for all x and that $J_0'(x) = -J_1(x)$.

9. Consider the Bessel equation of order ν

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \qquad x > 0,$$

where ν is real and positive.

(a) Show that $x = 0$ is a regular singular point and that the roots of the indicial equation are ν and $-\nu$.

(b) Corresponding to the larger root ν, show that one solution is

$$y_1(x) = x^{\nu}\left[1 - \frac{1}{1!(1+\nu)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(1+\nu)(2+\nu)}\left(\frac{x}{2}\right)^4 + \sum_{m=3}^{\infty}\frac{(-1)^m}{m!(1+\nu)\cdots(m+\nu)}\left(\frac{x}{2}\right)^{2m}\right].$$

(c) If 2ν is not an integer, show that a second solution is

$$y_2(x) = x^{-\nu}\left[1 - \frac{1}{1!(1-\nu)}\left(\frac{x}{2}\right)^2 + \frac{1}{2!(1-\nu)(2-\nu)}\left(\frac{x}{2}\right)^4 + \sum_{m=3}^{\infty}\frac{(-1)^m}{m!(1-\nu)\cdots(m-\nu)}\left(\frac{x}{2}\right)^{2m}\right].$$

Note that $y_1(x) \to 0$ as $x \to 0$, and that $y_2(x)$ is unbounded as $x \to 0$.

(d) Verify by direct methods that the power series in the expressions for $y_1(x)$ and $y_2(x)$ converge absolutely for all x. Also verify that y_2 is a solution, provided only that ν is not an integer.

10. In this section we showed that one solution of Bessel's equation of order zero

$$L[y] = x^2y'' + xy' + x^2y = 0$$

is J_0, where $J_0(x)$ is given by Eq. (7) with $a_0 = 1$. According to Theorem 5.6.1, a second solution has the form $(x > 0)$

$$y_2(x) = J_0(x)\ln x + \sum_{n=1}^{\infty} b_n x^n.$$

(a) Show that

$$L[y_2](x) = \sum_{n=2}^{\infty} n(n-1)b_n x^n + \sum_{n=1}^{\infty} nb_n x^n + \sum_{n=1}^{\infty} b_n x^{n+2} + 2xJ_0'(x). \qquad \text{(i)}$$

PROBLEMS

In each of Problems 1 through 4, sketch the graph of the given function. In each case determine whether f is continuous, piecewise continuous, or neither on the interval $0 \le t \le 3$.

1. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 2+t, & 1 < t \le 2 \\ 6-t, & 2 < t \le 3 \end{cases}$

2. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ (t-1)^{-1}, & 1 < t \le 2 \\ 1, & 2 < t \le 3 \end{cases}$

3. $f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 1, & 1 < t \le 2 \\ 3-t, & 2 < t \le 3 \end{cases}$

4. $f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 3-t, & 1 < t \le 2 \\ 1, & 2 < t \le 3 \end{cases}$

5. Find the Laplace transform of each of the following functions:
   (a) $f(t) = t$
   (b) $f(t) = t^2$
   (c) $f(t) = t^n$, where n is a positive integer
6. Find the Laplace transform of $f(t) = \cos at$, where a is a real constant.

Recall that $\cosh bt = (e^{bt} + e^{-bt})/2$ and $\sinh bt = (e^{bt} - e^{-bt})/2$. In each of Problems 7 through 10, find the Laplace transform of the given function; a and b are real constants.

7. $f(t) = \cosh bt$
8. $f(t) = \sinh bt$
9. $f(t) = e^{at} \cosh bt$
10. $f(t) = e^{at} \sinh bt$

Recall that $\cos bt = (e^{ibt} + e^{-ibt})/2$ and that $\sin bt = (e^{ibt} - e^{-ibt})/2i$. In each of Problems 11 through 14, find the Laplace transform of the given function; a and b are real constants. Assume that the necessary elementary integration formulas extend to this case.

11. $f(t) = \sin bt$
12. $f(t) = \cos bt$
13. $f(t) = e^{at} \sin bt$
14. $f(t) = e^{at} \cos bt$

In each of Problems 15 through 20, use integration by parts to find the Laplace transform of the given function; n is a positive integer and a is a real constant.

15. $f(t) = te^{at}$
16. $f(t) = t \sin at$
17. $f(t) = t \cosh at$
18. $f(t) = t^n e^{at}$
19. $f(t) = t^2 \sin at$
20. $f(t) = t^2 \sinh at$

In each of Problems 21 through 24, find the Laplace transform of the given function.

21. $f(t) = \begin{cases} 1, & 0 \le t < \pi \\ 0, & \pi \le t < \infty \end{cases}$

22. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & 1 \le t < \infty \end{cases}$

23. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & 1 \le t < \infty \end{cases}$

24. $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 2-t, & 1 \le t < 2 \\ 0, & 2 \le t < \infty \end{cases}$

In each of Problems 25 through 28, determine whether the given integral converges or diverges.

25. $\displaystyle\int_0^\infty (t^2+1)^{-1}\, dt$
26. $\displaystyle\int_0^\infty te^{-t}\, dt$
27. $\displaystyle\int_1^\infty t^{-2} e^t\, dt$
28. $\displaystyle\int_0^\infty e^{-t} \cos t\, dt$

PROBLEMS

In each of Problems 1 through 6, sketch the graph of the given function on the interval $t \geq 0$.

1. $g(t) = u_1(t) + 2u_3(t) - 6u_4(t)$
2. $g(t) = (t-3)u_2(t) - (t-2)u_3(t)$
3. $g(t) = f(t-\pi)u_\pi(t)$, where $f(t) = t^2$
4. $g(t) = f(t-3)u_3(t)$, where $f(t) = \sin t$
5. $g(t) = f(t-1)u_2(t)$, where $f(t) = 2t$
6. $g(t) = (t-1)u_1(t) - 2(t-2)u_2(t) + (t-3)u_3(t)$

In each of Problems 7 through 12:

(a) Sketch the graph of the given function.

(b) Express $f(t)$ in terms of the unit step function $u_c(t)$.

7. $f(t) = \begin{cases} 0, & 0 \leq t < 3, \\ -2, & 3 \leq t < 5, \\ 2, & 5 \leq t < 7, \\ 1, & t \geq 7. \end{cases}$
8. $f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ -1, & 1 \leq t < 2, \\ 1, & 2 \leq t < 3, \\ -1, & 3 \leq t < 4, \\ 0, & t \geq 4. \end{cases}$
9. $f(t) = \begin{cases} 1, & 0 \leq t < 2, \\ e^{-(t-2)}, & t \geq 2. \end{cases}$
10. $f(t) = \begin{cases} t^2, & 0 \leq t < 2, \\ 1, & t \geq 2. \end{cases}$
11. $f(t) = \begin{cases} t, & 0 \leq t < 1, \\ t-1, & 1 \leq t < 2, \\ t-2, & 2 \leq t < 3, \\ 0, & t \geq 3. \end{cases}$
12. $f(t) = \begin{cases} t, & 0 \leq t < 2, \\ 2, & 2 \leq t < 5, \\ 7-t, & 5 \leq t < 7, \\ 0, & t \geq 7. \end{cases}$

In each of Problems 13 through 18, find the Laplace transform of the given function.

13. $f(t) = \begin{cases} 0, & t < 2 \\ (t-2)^2, & t \geq 2 \end{cases}$
14. $f(t) = \begin{cases} 0, & t < 1 \\ t^2 - 2t + 2, & t \geq 1 \end{cases}$
15. $f(t) = \begin{cases} 0, & t < \pi \\ t - \pi, & \pi \leq t < 2\pi \\ 0, & t \geq 2\pi \end{cases}$
16. $f(t) = u_1(t) + 2u_3(t) - 6u_4(t)$
17. $f(t) = (t-3)u_2(t) - (t-2)u_3(t)$
18. $f(t) = t - u_1(t)(t-1), \qquad t \geq 0$

In each of Problems 19 through 24, find the inverse Laplace transform of the given function.

19. $F(s) = \dfrac{3!}{(s-2)^4}$
20. $F(s) = \dfrac{e^{-2s}}{s^2 + s - 2}$
21. $F(s) = \dfrac{2(s-1)e^{-2s}}{s^2 - 2s + 2}$
22. $F(s) = \dfrac{2e^{-2s}}{s^2 - 4}$
23. $F(s) = \dfrac{(s-2)e^{-s}}{s^2 - 4s + 3}$
24. $F(s) = \dfrac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$
25. Suppose that $F(s) = \mathcal{L}\{f(t)\}$ exists for $s > a \geq 0$.

    (a) Show that if c is a positive constant, then

$$\mathcal{L}\{f(ct)\} = \frac{1}{c}F\left(\frac{s}{c}\right), \qquad s > ca.$$

(b) Show that if k is a positive constant, then

$$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k} f\left(\frac{t}{k}\right).$$

(c) Show that if a and b are constants with $a > 0$, then

$$\mathcal{L}^{-1}\{F(as+b)\} = \frac{1}{a} e^{-bt/a} f\left(\frac{t}{a}\right).$$

In each of Problems 26 through 29, use the results of Problem 25 to find the inverse Laplace transform of the given function.

26. $F(s) = \dfrac{2^{n+1} n!}{s^{n+1}}$

27. $F(s) = \dfrac{2s+1}{4s^2+4s+5}$

28. $F(s) = \dfrac{1}{9s^2 - 12s + 3}$

29. $F(s) = \dfrac{e^2 e^{-4s}}{2s-1}$

In each of Problems 30 through 33, find the Laplace transform of the given function. In Problem 33, assume that term-by-term integration of the infinite series is permissible.

30. $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

31. $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \\ 1, & 2 \le t < 3 \\ 0, & t \ge 3 \end{cases}$

32. $f(t) = 1 - u_1(t) + \cdots + u_{2n}(t) - u_{2n+1}(t) = 1 + \displaystyle\sum_{k=1}^{2n+1} (-1)^k u_k(t)$

33. $f(t) = 1 + \displaystyle\sum_{k=1}^{\infty} (-1)^k u_k(t).$ See Figure 6.3.7.

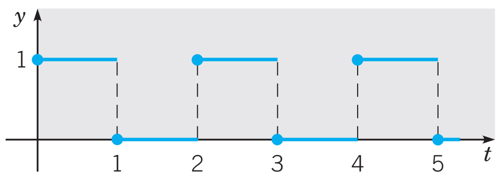

FIGURE 6.3.7 The function $f(t)$ in Problem 33; a square wave.

34. Let f satisfy $f(t+T) = f(t)$ for all $t \ge 0$ and for some fixed positive number T; f is said to be periodic with period T on $0 \le t < \infty$. Show that

$$\mathcal{L}\{f(t)\} = \frac{\displaystyle\int_0^T e^{-st} f(t)\, dt}{1 - e^{-sT}}.$$

In each of Problems 35 through 38, use the result of Problem 34 to find the Laplace transform of the given function.

35. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & 1 \le t < 2; \end{cases}$

$f(t+2) = f(t).$

Compare with Problem 33.

36. $f(t) = \begin{cases} 1, & 0 \le t < 1, \\ -1, & 1 \le t < 2; \end{cases}$

$f(t+2) = f(t).$

See Figure 6.3.8.