

7. $4y'' + 17y' + 4y = 0$

8. $16y'' + 24y' + 9y = 0$

9. $25y'' - 20y' + 4y = 0$

10. $2y'' + 2y' + y = 0$

In each of Problems 11 through 14, solve the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing t .

11. $9y'' - 12y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = -1$

12. $y'' - 6y' + 9y = 0, \quad y(0) = 0, \quad y'(0) = 2$

13. $9y'' + 6y' + 82y = 0, \quad y(0) = -1, \quad y'(0) = 2$

14. $y'' + 4y' + 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 1$



15. Consider the initial value problem

$$4y'' + 12y' + 9y = 0, \quad y(0) = 1, \quad y'(0) = -4.$$

(a) Solve the initial value problem and plot its solution for $0 \leq t \leq 5$.

(b) Determine where the solution has the value zero.

(c) Determine the coordinates (t_0, y_0) of the minimum point.

(d) Change the second initial condition to $y'(0) = b$ and find the solution as a function of b . Then find the critical value of b that separates solutions that always remain positive from those that eventually become negative.

16. Consider the following modification of the initial value problem in Example 2:

$$y'' - y' + 0.25y = 0, \quad y(0) = 2, \quad y'(0) = b.$$

Find the solution as a function of b , and then determine the critical value of b that separates solutions that grow positively from those that eventually grow negatively.



17. Consider the initial value problem

$$4y'' + 4y' + y = 0, \quad y(0) = 1, \quad y'(0) = 2.$$

(a) Solve the initial value problem and plot the solution.

(b) Determine the coordinates (t_M, y_M) of the maximum point.

(c) Change the second initial condition to $y'(0) = b > 0$ and find the solution as a function of b .

(d) Find the coordinates (t_M, y_M) of the maximum point in terms of b . Describe the dependence of t_M and y_M on b as b increases.

18. Consider the initial value problem

$$9y'' + 12y' + 4y = 0, \quad y(0) = a > 0, \quad y'(0) = -1.$$

(a) Solve the initial value problem.

(b) Find the critical value of a that separates solutions that become negative from those that are always positive.

19. Consider the equation $ay'' + by' + cy = 0$. If the roots of the corresponding characteristic equation are real, show that a solution to the differential equation either is everywhere zero or else can take on the value zero at most once.

Problems 20 through 22 indicate other ways of finding the second solution when the characteristic equation has repeated roots.

20. (a) Consider the equation $y'' + 2ay' + a^2y = 0$. Show that the roots of the characteristic equation are $r_1 = r_2 = -a$, so that one solution of the equation is e^{-at} .

(b) Use Abel's formula [Eq. (23) of Section 3.2] to show that the Wronskian of any two solutions of the given equation is

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = c_1 e^{-2at},$$

where c_1 is a constant.

(c) Let $y_1(t) = e^{-at}$ and use the result of part (b) to obtain a differential equation satisfied by a second solution $y_2(t)$. By solving this equation, show that $y_2(t) = te^{-at}$.

21. Suppose that r_1 and r_2 are roots of $ar^2 + br + c = 0$ and that $r_1 \neq r_2$; then $\exp(r_1 t)$ and $\exp(r_2 t)$ are solutions of the differential equation $ay'' + by' + cy = 0$. Show that $\phi(t; r_1, r_2) = [\exp(r_2 t) - \exp(r_1 t)]/(r_2 - r_1)$ is also a solution of the equation for $r_2 \neq r_1$. Then think of r_1 as fixed, and use L'Hôpital's rule to evaluate the limit of $\phi(t; r_1, r_2)$ as $r_2 \rightarrow r_1$, thereby obtaining the second solution in the case of equal roots.
22. (a) If $ar^2 + br + c = 0$ has equal roots r_1 , show that

$$L[e^{rt}] = a(e^{rt})'' + b(e^{rt})' + ce^{rt} = a(r - r_1)^2 e^{rt}. \quad (\text{i})$$

Since the right side of Eq. (i) is zero when $r = r_1$, it follows that $\exp(r_1 t)$ is a solution of $L[y] = ay'' + by' + cy = 0$.

(b) Differentiate Eq. (i) with respect to r , and interchange differentiation with respect to r and with respect to t , thus showing that

$$\frac{\partial}{\partial r} L[e^{rt}] = L \left[\frac{\partial}{\partial r} e^{rt} \right] = L[te^{rt}] = ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1). \quad (\text{ii})$$

Since the right side of Eq. (ii) is zero when $r = r_1$, conclude that $t \exp(r_1 t)$ is also a solution of $L[y] = 0$.

In each of Problems 23 through 30, use the method of reduction of order to find a second solution of the given differential equation.

23. $t^2 y'' - 4ty' + 6y = 0, \quad t > 0; \quad y_1(t) = t^2$
 24. $t^2 y'' + 2ty' - 2y = 0, \quad t > 0; \quad y_1(t) = t$
 25. $t^2 y'' + 3ty' + y = 0, \quad t > 0; \quad y_1(t) = t^{-1}$
 26. $t^2 y'' - t(t+2)y' + (t+2)y = 0, \quad t > 0; \quad y_1(t) = t$
 27. $xy'' - y' + 4x^3 y = 0, \quad x > 0; \quad y_1(x) = \sin x^2$
 28. $(x-1)y'' - xy' + y = 0, \quad x > 1; \quad y_1(x) = e^x$
 29. $x^2 y'' - (x - 0.1875)y = 0, \quad x > 0; \quad y_1(x) = x^{1/4} e^{2\sqrt{x}}$
 30. $x^2 y'' + xy' + (x^2 - 0.25)y = 0, \quad x > 0; \quad y_1(x) = x^{-1/2} \sin x$

31. The differential equation

$$y'' + \delta(xy' + y) = 0$$

arises in the study of the turbulent flow of a uniform stream past a circular cylinder. Verify that $y_1(x) = \exp(-\delta x^2/2)$ is one solution, and then find the general solution in the form of an integral.

32. The method of Problem 20 can be extended to second order equations with variable coefficients. If y_1 is a known nonvanishing solution of $y'' + p(t)y' + q(t)y = 0$, show that a second solution y_2 satisfies $(y_2/y_1)' = W(y_1, y_2)/y_1^2$, where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2 . Then use Abel's formula [Eq. (23) of Section 3.2] to determine y_2 .

In each of Problems 33 through 36, use the method of Problem 32 to find a second independent solution of the given equation.

33. $t^2y'' + 3ty' + y = 0, \quad t > 0; \quad y_1(t) = t^{-1}$
 34. $ty'' - y' + 4t^3y = 0, \quad t > 0; \quad y_1(t) = \sin(t^2)$
 35. $(x-1)y'' - xy' + y = 0, \quad x > 1; \quad y_1(x) = e^x$
 36. $x^2y'' + xy' + (x^2 - 0.25)y = 0, \quad x > 0; \quad y_1(x) = x^{-1/2} \sin x$

Behavior of Solutions as $t \rightarrow \infty$. Problems 37 through 39 are concerned with the behavior of solutions as $t \rightarrow \infty$.

37. If a, b , and c are positive constants, show that all solutions of $ay'' + by' + cy = 0$ approach zero as $t \rightarrow \infty$.
 38. (a) If $a > 0$ and $c > 0$, but $b = 0$, show that the result of Problem 37 is no longer true, but that all solutions are bounded as $t \rightarrow \infty$.
 (b) If $a > 0$ and $b > 0$, but $c = 0$, show that the result of Problem 37 is no longer true, but that all solutions approach a constant that depends on the initial conditions as $t \rightarrow \infty$. Determine this constant for the initial conditions $y(0) = y_0, y'(0) = y'_0$.
 39. Show that $y = \sin t$ is a solution of

$$y'' + (k \sin^2 t)y' + (1 - k \cos t \sin t)y = 0$$

for any value of the constant k . If $0 < k < 2$, show that $1 - k \cos t \sin t > 0$ and $k \sin^2 t \geq 0$. Thus observe that even though the coefficients of this variable-coefficient differential equation are nonnegative (and the coefficient of y' is zero only at the points $t = 0, \pi, 2\pi, \dots$), it has a solution that does not approach zero as $t \rightarrow \infty$. Compare this situation with the result of Problem 37. Thus we observe a not unusual situation in the study of differential equations: equations that are apparently very similar can have quite different properties.

Euler Equations. In each of Problems 40 through 45, use the substitution introduced in Problem 34 in Section 3.3 to solve the given differential equation.

40. $t^2y'' - 3ty' + 4y = 0, \quad t > 0$
 41. $t^2y'' + 2ty' + 0.25y = 0, \quad t > 0$
 42. $2t^2y'' - 5ty' + 5y = 0, \quad t > 0$
 43. $t^2y'' + 3ty' + y = 0, \quad t > 0$
 44. $4t^2y'' - 8ty' + 9y = 0, \quad t > 0$
 45. $t^2y'' + 5ty' + 13y = 0, \quad t > 0$

3.5 Nonhomogeneous Equations; Method of Undetermined Coefficients

We now return to the nonhomogeneous equation

$$L[y] = y'' + p(t)y' + q(t)y = g(t), \quad (1)$$

where p, q , and g are given (continuous) functions on the open interval I . The equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \quad (2)$$

and we should choose

$$Y(t) = e^{(\alpha+i\beta)t}(A_0t^n + \cdots + A_n) + e^{(\alpha-i\beta)t}(B_0t^n + \cdots + B_n),$$

or, equivalently,

$$Y(t) = e^{\alpha t}(A_0t^n + \cdots + A_n) \cos \beta t + e^{\alpha t}(B_0t^n + \cdots + B_n) \sin \beta t.$$

Usually, the latter form is preferred. If $\alpha \pm i\beta$ satisfy the characteristic equation corresponding to the homogeneous equation, we must, of course, multiply each of the polynomials by t to increase their degrees by one.

If the nonhomogeneous function involves both $\cos \beta t$ and $\sin \beta t$, it is usually convenient to treat these terms together, since each one individually may give rise to the same form for a particular solution. For example, if $g(t) = t \sin t + 2 \cos t$, the form for $Y(t)$ would be

$$Y(t) = (A_0t + A_1) \sin t + (B_0t + B_1) \cos t,$$

provided that $\sin t$ and $\cos t$ are not solutions of the homogeneous equation.

PROBLEMS

In each of Problems 1 through 14, find the general solution of the given differential equation.

- | | |
|--------------------------------------------------------------------------|------------------------------------------------------|
| 1. $y'' - 2y' - 3y = 3e^{2t}$ | 2. $y'' + 2y' + 5y = 3 \sin 2t$ |
| 3. $y'' - y' - 2y = -2t + 4t^2$ | 4. $y'' + y' - 6y = 12e^{3t} + 12e^{-2t}$ |
| 5. $y'' - 2y' - 3y = -3te^{-t}$ | 6. $y'' + 2y' = 3 + 4 \sin 2t$ |
| 7. $y'' + 9y = t^2e^{3t} + 6$ | 8. $y'' + 2y' + y = 2e^{-t}$ |
| 9. $2y'' + 3y' + y = t^2 + 3 \sin t$ | 10. $y'' + y = 3 \sin 2t + t \cos 2t$ |
| 11. $u'' + \omega_0^2 u = \cos \omega t, \quad \omega^2 \neq \omega_0^2$ | 12. $u'' + \omega_0^2 u = \cos \omega_0 t$ |
| 13. $y'' + y' + 4y = 2 \sinh t$ | 14. $y'' - y' - 2y = \cosh 2t$ |
| <i>Hint: $\sinh t = (e^t - e^{-t})/2$</i> | <i>Hint: $\cosh t = (e^t + e^{-t})/2$</i> |







In each of Problems 15 through 20, find the solution of the given initial value problem.


15. $y'' + y' - 2y = 2t, \quad y(0) = 0, \quad y'(0) = 1$
16. $y'' + 4y = t^2 + 3e^t, \quad y(0) = 0, \quad y'(0) = 2$
17. $y'' - 2y' + y = te^t + 4, \quad y(0) = 1, \quad y'(0) = 1$
18. $y'' - 2y' - 3y = 3te^{2t}, \quad y(0) = 1, \quad y'(0) = 0$
19. $y'' + 4y = 3 \sin 2t, \quad y(0) = 2, \quad y'(0) = -1$
20. $y'' + 2y' + 5y = 4e^{-t} \cos 2t, \quad y(0) = 1, \quad y'(0) = 0$


In each of Problems 21 through 28:

(a) Determine a suitable form for $Y(t)$ if the method of undetermined coefficients is to be used.

(b) Use a computer algebra system to find a particular solution of the given equation.

-  21. $y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin 3t$
-  22. $y'' + y = t(1 + \sin t)$
-  23. $y'' - 5y' + 6y = e^t \cos 2t + e^{2t}(3t + 4) \sin t$
-  24. $y'' + 2y' + 2y = 3e^{-t} + 2e^{-t} \cos t + 4e^{-t}t^2 \sin t$
-  25. $y'' - 4y' + 4y = 2t^2 + 4te^{2t} + t \sin 2t$
-  26. $y'' + 4y = t^2 \sin 2t + (6t + 7) \cos 2t$

 27. $y'' + 3y' + 2y = e^t(t^2 + 1) \sin 2t + 3e^{-t} \cos t + 4e^t$

 28. $y'' + 2y' + 5y = 3te^{-t} \cos 2t - 2te^{-2t} \cos t$

29. Consider the equation

$$y'' - 3y' - 4y = 2e^{-t} \quad (i)$$

from Example 5. Recall that $y_1(t) = e^{-t}$ and $y_2(t) = e^{4t}$ are solutions of the corresponding homogeneous equation. Adapting the method of reduction of order (Section 3.4), seek a solution of the nonhomogeneous equation of the form $Y(t) = v(t)y_1(t) = v(t)e^{-t}$, where $v(t)$ is to be determined.

(a) Substitute $Y(t)$, $Y'(t)$, and $Y''(t)$ into Eq. (i) and show that $v(t)$ must satisfy $v'' - 5v' = 2$.

(b) Let $w(t) = v'(t)$ and show that $w(t)$ must satisfy $w' - 5w = 2$. Solve this equation for $w(t)$.

(c) Integrate $w(t)$ to find $v(t)$ and then show that


$$Y(t) = -\frac{2}{5}te^{-t} + \frac{1}{5}c_1e^{4t} + c_2e^{-t}.$$

The first term on the right side is the desired particular solution of the nonhomogeneous equation. Note that it is a product of t and e^{-t} .

30. Determine the general solution of

$$y'' + \lambda^2 y = \sum_{m=1}^N a_m \sin m\pi t,$$

where $\lambda > 0$ and $\lambda \neq m\pi$ for $m = 1, \dots, N$.

 31. In many physical problems the nonhomogeneous term may be specified by different formulas in different time periods. As an example, determine the solution $y = \phi(t)$ of

$$y'' + y = \begin{cases} t, & 0 \leq t \leq \pi, \\ \pi e^{\pi-t}, & t > \pi, \end{cases}$$

satisfying the initial conditions $y(0) = 0$ and $y'(0) = 1$. Assume that y and y' are also continuous at $t = \pi$. Plot the nonhomogeneous term and the solution as functions of time. *Hint:* First solve the initial value problem for $t \leq \pi$; then solve for $t > \pi$, determining the constants in the latter solution from the continuity conditions at $t = \pi$.

 32. Follow the instructions in Problem 31 to solve the differential equation

$$y'' + 2y' + 5y = \begin{cases} 1, & 0 \leq t \leq \pi/2, \\ 0, & t > \pi/2 \end{cases}$$

with the initial conditions $y(0) = 0$ and $y'(0) = 0$.

Behavior of Solutions as $t \rightarrow \infty$. In Problems 33 and 34, we continue the discussion started with Problems 37 through 39 of Section 3.4. Consider the differential equation

$$ay'' + by' + cy = g(t), \quad (i)$$

where a , b , and c are positive.

33. If $Y_1(t)$ and $Y_2(t)$ are solutions of Eq. (i), show that $Y_1(t) - Y_2(t) \rightarrow 0$ as $t \rightarrow \infty$. Is this result true if $b = 0$?

34. If $g(t) = d$, a constant, show that every solution of Eq. (i) approaches d/c as $t \rightarrow \infty$. What happens if $c = 0$? What if $b = 0$ also?