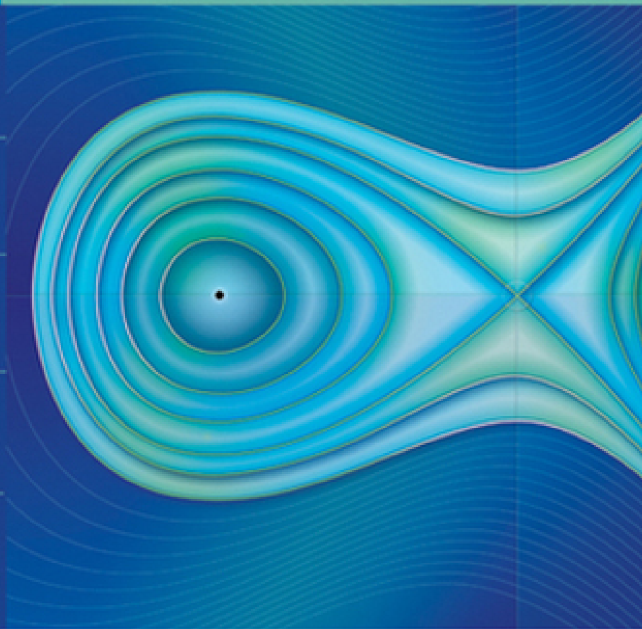
10th Edition

# Elementary Differential Equations and Boundary Value Problems



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## PROBLEMS

In each of Problems 1 through 6:

(a) Express the general solution of the given system of equations in terms of real-valued functions.

(b) Also draw a direction field, sketch a few of the trajectories, and describe the behavior of the solutions as $t \to \infty$.

1. $\mathbf{x}' = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \mathbf{x}$
2. $\mathbf{x}' = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix} \mathbf{x}$
3. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \mathbf{x}$
4. $\mathbf{x}' = \begin{pmatrix} 2 & -\frac{5}{2} \\ \frac{9}{5} & -1 \end{pmatrix} \mathbf{x}$
5. $\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \mathbf{x}$
6. $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \mathbf{x}$

In each of Problems 7 and 8, express the general solution of the given system of equations in terms of real-valued functions.

7. $\mathbf{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix} \mathbf{x}$
8. $\mathbf{x}' = \begin{pmatrix} -3 & 0 & 2 \\ 1 & -1 & 0 \\ -2 & -1 & 0 \end{pmatrix} \mathbf{x}$

In each of Problems 9 and 10, find the solution of the given initial value problem. Describe the behavior of the solution as $t \to \infty$.

9. $\mathbf{x}' = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
10. $\mathbf{x}' = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

In each of Problems 11 and 12:

(a) Find the eigenvalues of the given system.

(b) Choose an initial point (other than the origin) and draw the corresponding trajectory in the $x_1x_2$-plane.

(c) For your trajectory in part (b), draw the graphs of $x_1$ versus $t$ and of $x_2$ versus $t$.

(d) For your trajectory in part (b), draw the corresponding graph in three-dimensional $tx_1x_2$-space.

11. $\mathbf{x}' = \begin{pmatrix} \frac{3}{4} & -2 \\ 1 & -\frac{5}{4} \end{pmatrix} \mathbf{x}$
12. $\mathbf{x}' = \begin{pmatrix} -\frac{4}{5} & 2 \\ -1 & \frac{6}{5} \end{pmatrix} \mathbf{x}$

In each of Problems 13 through 20, the coefficient matrix contains a parameter $\alpha$. In each of these problems:

(a) Determine the eigenvalues in terms of $\alpha$.

(b) Find the critical value or values of $\alpha$ where the qualitative nature of the phase portrait for the system changes.

(c) Draw a phase portrait for a value of $\alpha$ slightly below, and for another value slightly above, each critical value.

13. $\mathbf{x}' = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha \end{pmatrix} \mathbf{x}$
14. $\mathbf{x}' = \begin{pmatrix} 0 & -5 \\ 1 & \alpha \end{pmatrix} \mathbf{x}$
15. $\mathbf{x}' = \begin{pmatrix} 2 & -5 \\ \alpha & -2 \end{pmatrix} \mathbf{x}$
16. $\mathbf{x}' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \alpha & \frac{5}{4} \end{pmatrix} \mathbf{x}$

20. Consider the linear system

$$dx/dt = a_{11}x + a_{12}y, \qquad dy/dt = a_{21}x + a_{22}y,$$

where $a_{11}, a_{12}, a_{21}$, and $a_{22}$ are real constants. Let $p = a_{11} + a_{22}$, $q = a_{11}a_{22} - a_{12}a_{21}$, and $\Delta = p^2 - 4q$. Observe that $p$ and $q$ are the trace and determinant, respectively, of the coefficient matrix of the given system. Show that the critical point $(0, 0)$ is a

(a) Node if $q > 0$ and $\Delta \geq 0$; (b) Saddle point if $q < 0$;

(c) Spiral point if $p \neq 0$ and $\Delta < 0$; (d) Center if $p = 0$ and $q > 0$.

*Hint:* These conclusions can be reached by studying the eigenvalues $r_1$ and $r_2$. It may also be helpful to establish, and then to use, the relations $r_1r_2 = q$ and $r_1 + r_2 = p$.

21. Continuing Problem 20, show that the critical point $(0, 0)$ is

(a) Asymptotically stable if $q > 0$ and $p < 0$;

(b) Stable if $q > 0$ and $p = 0$;

(c) Unstable if $q < 0$ or $p > 0$.

The results of Problems 20 and 21 are summarized visually in Figure 9.1.9.

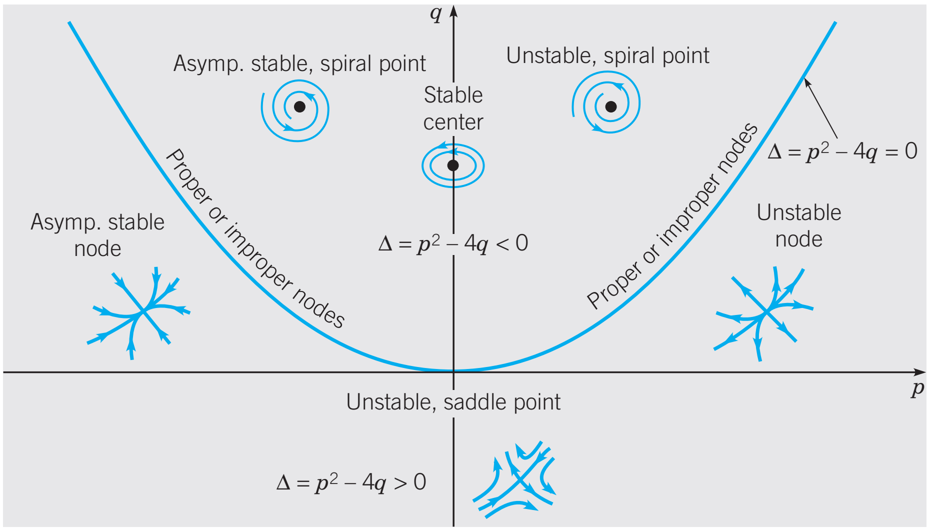


**FIGURE 9.1.9** Stability diagram.

22. In this problem we illustrate how a $2 \times 2$ system with eigenvalues $\lambda \pm i\mu$ can be transformed into the system (11). Consider the system in Problem 12:

$$\mathbf{x}' = \begin{pmatrix} 2 & -2.5 \\ 1.8 & -1 \end{pmatrix} \mathbf{x} = \mathbf{A}\mathbf{x}. \tag{i}$$

(a) Show that the eigenvalues of this system are $r_1 = 0.5 + 1.5i$ and $r_2 = 0.5 - 1.5i$.

(b) Show that the eigenvector corresponding to $r_1$ can be chosen as

$$\boldsymbol{\xi}^{(1)} = \begin{pmatrix} 5 \\ 3 - 3i \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} + i \begin{pmatrix} 0 \\ -3 \end{pmatrix}. \tag{ii}$$

All of the basins of attraction are congruent to the shaded one; the only difference is that they are translated horizontally by appropriate distances. Note that it is mathematically possible (but physically unrealizable) to choose initial conditions exactly on a separatrix so that the resulting motion leads to a balanced pendulum in a vertically upward position of unstable equilibrium.

An important difference between nonlinear autonomous systems and the linear systems discussed in Section 9.1 is illustrated by the pendulum equations. Recall that the linear system (1) has only the single critical point $\mathbf{x} = \mathbf{0}$ if $\det \mathbf{A} \neq 0$. Thus, if the origin is asymptotically stable, then not only do trajectories that start close to the origin approach it, but, in fact, every trajectory approaches the origin. In this case the critical point $\mathbf{x} = \mathbf{0}$ is said to be **globally asymptotically stable**. This property of linear systems is not, in general, true for nonlinear systems, even if the nonlinear system has only one asymptotically stable critical point. Therefore, for nonlinear systems, it is important to determine (or to estimate) the basin of attraction for each asymptotically stable critical point.

## PROBLEMS

In each of Problems 1 through 4, verify that $(0, 0)$ is a critical point, show that the system is locally linear, and discuss the type and stability of the critical point $(0, 0)$ by examining the corresponding linear system.

1. $dx/dt = x - y^2, \qquad dy/dt = x - 2y + x^2$
2. $dx/dt = -x + y + 2xy, \qquad dy/dt = -4x - y + x^2 - y^2$
3. $dx/dt = (1 + x) \sin y, \qquad dy/dt = 1 - x - \cos y$
4. $dx/dt = x + y^2, \qquad dy/dt = x + y$

In each of Problems 5 through 18:

(a) Determine all critical points of the given system of equations.

(b) Find the corresponding linear system near each critical point.

(c) Find the eigenvalues of each linear system. What conclusions can you then draw about the nonlinear system?

(d) Draw a phase portrait of the nonlinear system to confirm your conclusions, or to extend them in those cases where the linear system does not provide definite information about the nonlinear system.

5. $dx/dt = (2 + x)(y - x), \qquad dy/dt = (4 - x)(y + x)$
6. $dx/dt = x - x^2 - xy, \qquad dy/dt = 3y - xy - 2y^2$
7. $dx/dt = 1 - y, \qquad dy/dt = x^2 - y^2$
8. $dx/dt = x - x^2 - xy, \qquad dy/dt = \frac{1}{2}y - \frac{1}{4}y^2 - \frac{3}{4}xy$
9. $dx/dt = (2 + y)(y - 0.5x), \qquad dy/dt = (2 - x)(y + 0.5x)$
10. $dx/dt = x + x^2 + y^2, \qquad dy/dt = y - xy$
11. $dx/dt = 2x + y + xy^3, \qquad dy/dt = x - 2y - xy$
12. $dx/dt = (1 + x) \sin y, \qquad dy/dt = 1 - x - \cos y$
13. $dx/dt = x - y^2, \qquad dy/dt = y - x^2$
14. $dx/dt = 1 - xy, \qquad dy/dt = x - y^3$
15. $dx/dt = -2x - y - x(x^2 + y^2), \qquad dy/dt = x - y + y(x^2 + y^2)$
16. $dx/dt = y + x(1 - x^2 - y^2), \qquad dy/dt = -x + y(1 - x^2 - y^2)$

whether there is a region containing the origin in the $uv$-plane where the derivative $\dot{V}$ with respect to the system (20) is negative definite. We compute $\dot{V}(u, v)$ and find that

$$\begin{aligned}\dot{V}(u, v) &= V_u \frac{du}{dt} + V_v \frac{dv}{dt} \\ &= 2u(-0.5u - 0.5v - u^2 - uv) + 2v(-0.25u - 0.5v - 0.5uv - v^2),\end{aligned}$$

or

$$\dot{V}(u, v) = -\left[(u^2 + 1.5uv + v^2) + (2u^3 + 2u^2v + uv^2 + 2v^3)\right], \tag{21}$$

where we have collected together the quadratic and cubic terms. We want to show that the expression in square brackets in Eq. (21) is positive definite, at least for $u$ and $v$ sufficiently small. Observe that the quadratic terms can be written as

$$u^2 + 1.5uv + v^2 = 0.25(u^2 + v^2) + 0.75(u + v)^2, \tag{22}$$

so these terms are positive definite. On the other hand, the cubic terms in Eq. (21) may be of either sign. Thus we must show that, in some neighborhood of $u = 0$, $v = 0$, the cubic terms are smaller in magnitude than the quadratic terms; that is,

$$\left|2u^3 + 2u^2v + uv^2 + 2v^3\right| < 0.25(u^2 + v^2) + 0.75(u + v)^2. \tag{23}$$

To estimate the left side of Eq. (23), we introduce polar coordinates $u = r\cos\theta$, $v = r\sin\theta$. Then

$$\begin{aligned}\left|2u^3 + 2u^2v + uv^2 + 2v^3\right| &= r^3\left|2\cos^3\theta + 2\cos^2\theta\sin\theta + \cos\theta\sin^2\theta + 2\sin^3\theta\right| \\ &\le r^3\left[2|\cos^3\theta| + 2\cos^2\theta|\sin\theta| + |\cos\theta|\sin^2\theta + 2|\sin^3\theta|\right] \\ &\le 7r^3,\end{aligned}$$

since $|\sin\theta|, |\cos\theta| \le 1$. To satisfy Eq. (23), it is now certainly sufficient to satisfy the more stringent requirement

$$7r^3 < 0.25(u^2 + v^2) = 0.25r^2 = \tfrac{1}{4}r^2,$$

which yields $r < 1/28$. Thus, at least in this disk, the hypotheses of Theorem 9.6.1 are satisfied, so the origin is an asymptotically stable critical point of the system (20). The same is then true of the critical point $(0.5, 0.5)$ of the original system (18).

If we refer to Theorem 9.6.3, the preceding argument also shows that the disk with center $(0.5, 0.5)$ and radius $1/28$ is a region of asymptotic stability for the system (18). This is a severe underestimate of the full basin of attraction, as the discussion in Section 9.4 shows. To obtain a better estimate of the actual basin of attraction from Theorem 9.6.3, we would have to estimate the terms in Eq. (23) more accurately, or use a better (and presumably more complicated) Liapunov function, or both.

## PROBLEMS

In each of Problems 1 through 4, construct a suitable Liapunov function of the form $ax^2 + cy^2$, where $a$ and $c$ are to be determined. Then show that the critical point at the origin is of the indicated type.

1. $dx/dt = -x^3 + xy^2, \quad dy/dt = -2x^2y - y^3;$ asymptotically stable
2. $dx/dt = -\frac{1}{2}x^3 + 2xy^2, \quad dy/dt = -y^3;$ asymptotically stable
3. $dx/dt = -x^3 + 2y^3, \quad dy/dt = -2xy^2;$ stable (at least)
4. $dx/dt = x^3 - y^3, \quad dy/dt = 2xy^2 + 4x^2y + 2y^3;$ unstable