MATH2050B Mathematical Analysis I

Make-up Test 1 suggested Solution*

Question 1. State the definitions/notations, and the negation for (v) and (vi).

(i) Archimedean properties of $\mathbb{N}$ in $\mathbb{R}$.

(ii) $x \in \mathbb{R}$ is a lower bound of B.

(iii) $\lim_n x_n = \ell$.

(iv) $\lim_n x_n = -\infty$.

(v) (x_n) is Cauchy.

(vi) A is order-convex (convex).

Solution:

(i) **Archimedean Property of $\mathbb{N}$:** Let $r \in \mathbb{R}$. Then there exists $n \in \mathbb{N}$ such that $r < n$.

(ii) We say x is a lower bound of B, if $x \leq b$ for all $b \in B$.

(iii) For every $\varepsilon > 0$ there exists a natural number $K(\varepsilon)$ such that for all $n \geq K(\varepsilon)$, the terms x_n satisfy $|x_n - \ell| < \varepsilon$.

(iv) For every $r \in \mathbb{R}$, there is an $N(r)$ such that for every $n \geq N(r), x_n < r$.

(v) (x_n) is said to be Cauchy if for any $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for any $m, n \geq N_0$,

$$|x_n - x_m| < \varepsilon.$$

Negation: There exists $\varepsilon_0 > 0$ such that for every H there exist at least one $n > H$ and at least one $m > H$ such that $|x_n - x_m| \geq \varepsilon_0$.

(vi) $A \subseteq \mathbb{R}$ is said to be oder-convex if, for any $a_1, a_2 \in A$ and any $z \in \mathbb{R}$ with $a_1 < z < a_2$, one has $z \in A$.

Negation: There exists two points $x, y \in A$, and a point $z_0 = t_0 x + (1 - t_0) y$ with $0 < t_0 < 1$, such that $z_0 \notin A$.

Question 2. State the following results/theorems:

*please kindly send an email to cyma@math.cuhk.edu.hk if you have any question.

(i) Characterization theorem for intervals.

(ii) The nested intervals theorem.

(iii) Bolzano–Weierstrass Theorem.

(iv) Cauchy criterion.

Solution:

(i) **Characterization of Intervals**: Let $I \subseteq \mathbb{R}$ be order-convex. Then I is an interval.

(ii) **The nested intervals theorem:** Let $I_n := [a_n, b_n] \subseteq \mathbb{R}$ with $a_n \leq b_n$ be such that

$$I_{n+1} \subseteq I_n, \quad \forall n \in \mathbb{N}.$$

Then

$$\bigcap_{n \in \mathbb{N}} I_n \neq \varnothing.$$

(iii) **Bolzano-Weierstrass Theorem:** A bounded sequence of real numbers has a convergent subsequence.

(iv) **Cauchy Convergence Criterion:** A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

Question 3. *(Not to use on any theorem (limits))* In the terminology of $\varepsilon - N$, show that

(i) If $\lim_n x_n = x, \lim_n y_n = y$ in $\mathbb{R}$ and

$$x_n \leqslant y_n \quad \forall n \geqslant 1997,$$

then $x \leqslant y$.

(ii) If $\lim_n z_n = \ell > 0$ then $\exists N \in \mathbb{N}$ with $N \geqslant 2047$ s.t.

$$\frac{2\ell}{3} < z_n < 2\ell \quad \forall n \geqslant N.$$

(iii) Suppose $\lim_n x_n = 5, \lim_n y_n = 2$. Then

$$\lim_n \frac{x_n}{y_n} = \frac{5}{2}.$$

(Hint: What wonld you do if 5,2 are x, y?)

(iv) Let (x_n) be a $\downarrow$ (decreasing) sequence of positive real numbers. Show $\lim\limits_n x_n$ exists in $\mathbb{R}$.

Solution:

(i) For any $\varepsilon > 0$, there exists $N_1(\varepsilon) \in \mathbb{N}$ so that for any $n \geq N_1(\varepsilon)$,

$$|x_n - x| < \varepsilon/2.$$

Since $\lim_n y_n = y$ there exists $N_2(\varepsilon) \in \mathbb{N}$ so that for any $n \geq N_2(\varepsilon)$,

$$|y_n - y| < \varepsilon/2.$$

Let $N = \max\{N_1(\varepsilon), N_2(\varepsilon), 1997\}$, then for any $n \geq N$,

$$\begin{aligned} y - x &= (y - y_n) + (y_n - x_n) + (x_n - x) \\ &\geq -|y - y_n| + (y_n - x_n) - |x_n - x| \\ &> (y_n - x_n) - \varepsilon \\ &\geq -\varepsilon. \end{aligned}$$

Thus we have $y - x > -\varepsilon$. Since ε is arbitrary, we have $y - x \geq 0$, as desired.

(ii) Since $\lim_n z_n = \ell > 0$, there exists $N_1 \in \mathbb{N}$ such that for any $n \geq N_1$, we have

$$|z_n - \ell| < \frac{\ell}{3}, \qquad \text{i.e.} \qquad -\frac{\ell}{3} < z_n - \ell < \frac{\ell}{3}.$$

It follows that $\frac{2\ell}{3} < z_n < 2\ell$, for any $n \geq N_1$.

Let $N = \max\{N_1, 2047\}$, it is easily seen that $\frac{2\ell}{3} < z_n < 2\ell$, for any $n \geq N$.

(iii) Firstly we show that the product $(x_n y_n)$ is convergent with $\lim x_n y_n = 10$.

Since $\lim_n y_n = 2$, there exists $N_0 \in \mathbb{N}$ such that for any $n \geq N_0$, we have $|y_n - 2| < 1$. It directly follows that $|y_n| < 4$ for any $n \geq N_0$.

Fix $\varepsilon > 0$. Take $\varepsilon' > 0$ such that $\varepsilon' = \min\{\frac{\varepsilon}{9}, 1\}$. Since $\lim_n x_n = 5$, there exists $N_1(\varepsilon) \in \mathbb{N}$ such that for any $n \geq N_1(\varepsilon)$, we have $|x_n - 5| < \varepsilon'$. Similarly, since $\lim_n y_n = 2$, there exists $N_2(\varepsilon) \in \mathbb{N}$ such that for any $n \geq N_2(\varepsilon)$, we obtain $|y_n - 2| < \varepsilon'$.

Hence, the triangle inequality implies that

$$|x_n y_n - 10| \leq |x_n y_n - 5y_n| + |5y_n - 10| \leq |x_n - 5|\,|y_n| + |5|\,|y_n - 2| < \varepsilon' \cdot 4 + 5\varepsilon' = 9\varepsilon' \leq \varepsilon,$$

for all $n \geq \max\{N_0, N_1(\varepsilon), N_2(\varepsilon)\}$. This implies that $(x_n y_n)$ is convergent and $\lim x_n y_n = ab$.

For showing $\lim_n \frac{x_n}{y_n} = \frac{5}{2}$, it suffices to show that the sequence $\left(\dfrac{1}{y_n}\right)$ converges to $1/2$ by using above result.

Let $\varepsilon > 0$ be as above and $N = \max\{N_0(\varepsilon), N_1(\varepsilon), N_2(\varepsilon)\}$. It is noted that there is a positive integer $N_3 > N$ such that $|y_n - 2| < 1$ for all $n \geq N_3$. This gives $|y_n| > 1$ for all $n \geq N_3$. Hence, we have

$$\left|\frac{1}{y_n} - \frac{1}{2}\right| = \frac{|y_n - 2|}{|y_n|\,2} \leq \varepsilon/2,$$

for all $n \geq N_3$. The proof is complete.

(iv) Since x_n is bounded below by 0, by The Completeness Property of $\mathbb{R}$, the sequence has an infimum in $\mathbb{R}$. Suppose $\inf x_n = a$. Then for given $\varepsilon > 0$, there exists n_0 such that $a + \varepsilon \geq x_{n_0}$. Since (x_n) is decreasing, we have $x_{n_0} \geq x_n$ for all $n \geq n_0$. This implies that

$$a + \varepsilon \geq x_n \geq a \geq a - \varepsilon \quad \text{for all} \quad n \geq n_0.$$

That is $\lim_n x_n = a$.