MATH2050B Mathematical Analysis I

Homework 5 suggested Solution*

Question 1. Let $x_0 \in \mathbb{R}$, $\xi \in \mathbb{R} \cup \{\pm\infty\} = [-\infty, \infty]$ and $f : \mathbb{R} \setminus \{x_0\} \to \mathbb{R}$. Show that

$$\lim_{x \to x_0} f(x) = \xi \Longleftrightarrow \lim_{x \to x_0-} f(x) = \xi = \lim_{x \to x_0+} f(x),$$

and hence that $\lim\limits_{x \to x_0} f(x)$ not exist in $[-\infty, \infty]$ if $\lim\limits_{x \to x_0-} f(x) \neq \lim\limits_{x \to x_0+} f(x)$.

Solution: Case 1: $\xi \in \mathbb{R}$. Suppose that $\lim\limits_{x \to x_0} f(x) = \xi$. For any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for any $0 < |x - x_0| < \delta(\varepsilon)$, we have $|f(x) - \xi| < \varepsilon$. It follows that for any $y \in (x_0, x_0 + \delta(\varepsilon))$ and $z \in (x_0 - \delta(\varepsilon), x_0)$,

$$|f(y) - \xi| < \varepsilon \quad \text{and} \quad |f(z) - \xi| < \varepsilon,$$

which implies that $\lim\limits_{x \to x_0-} f(x) = \xi = \lim\limits_{x \to x_0+} f(x)$.

Conversely, suppose that $\lim\limits_{x \to x_0-} f(x) = \xi = \lim\limits_{x \to x_0+} f(x)$. For any $\varepsilon > 0$, there exists $\delta_1(\varepsilon) > 0$ and $\delta_2(\varepsilon) > 0$ such that for any $z \in (x_0 - \delta_1(\varepsilon), x_0)$ and $y \in (x_0, x_0 + \delta_2(\varepsilon))$,

$$|f(z) - \xi| < \varepsilon \quad \text{and} \quad |f(y) - \xi| < \varepsilon.$$

Let $\delta(\varepsilon) = \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}$, then for any $0 < |x - x_0| < \delta(\varepsilon)$, we have $|f(x) - \xi| < \varepsilon$. This yields that $\lim\limits_{x \to x_0} f(x) = \xi$.

Case 2: $\xi \in \{\pm\infty\}$. Assume that $\xi = +\infty$ *(The proof for $\xi = -\infty$ is similar)*. Suppose that $\lim\limits_{x \to x_0} f(x) = \xi$. For any $M \in \mathbb{R}$, there exists $\delta(M) > 0$ such that for any $0 < |x - x_0| < \delta(M)$, we have $f(x) > M$. It follows that for any $y \in (x_0, x_0 + \delta(M))$ and $z \in (x_0 - \delta(M), x_0)$,

$$f(y) > M \quad \text{and} \quad f(z) > M,$$

which implies that $\lim\limits_{x \to x_0-} f(x) = +\infty = \lim\limits_{x \to x_0+} f(x)$.

Conversely, suppose that $\lim\limits_{x \to x_0-} f(x) = \xi = \lim\limits_{x \to x_0+} f(x)$. For any $M \in \mathbb{R}$, there exists $\delta_1(M) > 0$ and $\delta_2(M) > 0$ such that for any $z \in (x_0 - \delta_1(M), x_0)$ and $y \in (x_0, x_0 + \delta_2(M))$,

$$f(z) > M \quad \text{and} \quad f(y) > M.$$

*please kindly send an email to cyma@math.cuhk.edu.hk if you have any question.

Let $\delta(M) = \min\{\delta_1(M), \delta_2(M)\}$, then for any $0 < |x - x_0| < \delta(M)$, we have $f(x) > M$. This yields that $\lim_{x \to x_0} f(x) = \xi$.

Question 2*. Use the definition of limit to show that

$$\lim_{x \to -1} \frac{x+5}{2x+3} = 4.$$

Solution: Let $\delta = \frac{1}{4}$. Note that if $|x - (-1)| < \delta$, then we have

$$2x + 3 > 2(\frac{-5}{4}) + 3 = \frac{1}{2} \quad \text{and} \quad 2x + 3 < 2(\frac{-3}{4}) + 3 = \frac{3}{2},$$

which implies that $|2x + 3| > \frac{1}{2}$.

Given any $\varepsilon > 0$, take $\delta' = \min\{\frac{\varepsilon}{14}, \delta\}$, then for any $|x + 1| < \delta'$, we have

$$\begin{aligned} \left| \frac{x+5}{2x+3} - 4 \right| &= \left| \frac{x + 5 - 4(2x+3)}{2x+3} \right| \\ &= \left| \frac{-7x-7}{2x+3} \right| \\ &< 7 \cdot \frac{|x+1|}{1/2} \\ &\le 14 \cdot \frac{\varepsilon}{14} = \varepsilon, \end{aligned}$$

which completes the proof.

Question 4*. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by setting $f(x) := x$ if x is rational, and $f(x) = 0$ if x is irrational.

(a) Show that f has a limit at $x = 0$.

(b) Use a sequential argument to show that if $c \neq 0$, then f does not have a limit at c.

Solution:

(a) For any $\varepsilon > 0$, take $\delta(\varepsilon) = \varepsilon/2$, then for any $|x - 0| < \delta(\varepsilon)$, we have

$$|f(x) - 0| \le |x| \le \delta(\varepsilon) < \varepsilon,$$

which implies that $\lim_{x \to 0} f(x) = 0$.

(b) Since both $\mathbb{Q}$ and $\mathbb{Q}^c$ are dense in $\mathbb{R}$, there exist sequence $\{x_n\} \subseteq \mathbb{Q}$ and $\{y_n\} \subseteq \mathbb{Q}^c$ such that

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = c.$$

However, it follows from the definition of function f that

$$f(y_n) = 0 \quad \text{and} \quad f(x_n) = x_n, \quad \text{for all } n \in \mathbb{N}.$$

Thus we have $\lim_{n\to\infty} f(y_n) = 0$ and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = c$. Since $c \neq 0$, we have $\lim_{n\to\infty} f(y_n) \neq \lim_{n\to\infty} f(x_n)$, hence that f does not have a limit at c.

Question 6*. Apply Theorem 4.2.4 to determine the following limits:

$$\lim_{x\to 1} \frac{x^2+2}{x^2-2} \quad (x > 0).$$

Solution: Let $A = [1 - \frac{1}{5}, 1 + \frac{1}{5}]$, then $x^2 - 2 \neq 0$ for any $x \in A$. Since $\lim_{x\to 1} x^2 - 2 = -1 (\neq 0)$, and $\lim_{x\to 1} x^2 + 2 = 3$, it follows from Theorem 4.2.4 that

$$\lim_{x\to 1} \frac{x^2+2}{x^2-2} = \frac{\lim_{x\to 1}(x^2+2)}{\lim_{x\to 1}(x^2-2)} = -3.$$

Question 7*. Determine the following limits and state which theorems are used in each case. (You may wish to use Exercise 15 below.)

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} \quad (x > 0).$$

Solution: Notice that

$$\frac{\sqrt{x}-1}{x-1} = \frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)} = \frac{1}{\sqrt{x}+1}.$$

It follows from Exercise 15 that $\lim_{x\to 1} \sqrt{x} = \sqrt{\lim_{x\to 1} x} = 1$. Let $A = [0, 2]$, then $\sqrt{x} + 1 \neq 0$ for all $x \in A$. It follows from Theorem 4.2.4 that

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{\lim_{x\to 1}(\sqrt{x}+1)} = \frac{1}{(\sqrt{\lim_{x\to 1} x} + 1)} = \frac{1}{2}.$$

Question 8*. Find $\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2}$ where $x > 0$.

(Quite difficult to check with $\varepsilon - \delta$! – not required at this time.)

Solution:

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\begin{aligned}
\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2} &= \lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1+3x}}{x(1+2x)} \\
&= \lim_{x\to 0} \frac{(\sqrt{1+2x}-\sqrt{1+3x})(\sqrt{1+2x}+\sqrt{1+3x})}{x(1+2x)(\sqrt{1+2x}+\sqrt{1+3x})} \\
&= \lim_{x\to 0} \frac{(1+2x)-(1+3x)}{x(1+2x)(\sqrt{1+2x}+\sqrt{1+3x})} \\
&= \lim_{x\to 0} \frac{-x}{x(1+2x)(\sqrt{1+2x}+\sqrt{1+3x})} \\
&= \lim_{x\to 0} \frac{-1}{(1+2x)(\sqrt{1+2x}+\sqrt{1+3x})} \\
&= \frac{-1}{\lim_{x\to 0}(1+2x)(\sqrt{1+2x}+\sqrt{1+3x})} \\
&= \frac{-1}{1\cdot(\sqrt{1+2\cdot 0}+\sqrt{1+3\cdot 0})} \\
&= -\frac{1}{2}.
\end{aligned}
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