# MATH2050B Mathematical Analysis I

## Homework 6 suggested Solution*

**Question 2***. Check Q1 is the $\varepsilon$ - $\delta$ terminology. Hint: Let $M \in \mathbb{R}$. Wish to find $\delta > 0$ s.t.

$$\frac{x}{x-1} > M, \text{ whenever } x \in (1, 1+\delta),$$

Is it equivalent to the following inequalities?

$$\begin{aligned} x > M(x-1) &= Mx - M \\ &\iff M > (M-1)x \\ &\iff \frac{M}{M-1} > x \quad (\text{Assume } M > 1) \\ &\iff \frac{M-1+1}{M-1} > x \\ &\iff 1 + \frac{1}{M-1} > x. \end{aligned}$$

Thus we may take $\delta := \frac{1}{M-1}$ (with $M > 1$).

**Solution:** For any $M > 1$, take $\delta = \frac{1}{M-1}$. For any $x \in (1, 1+\delta)$,

$$\begin{aligned} \frac{x}{x-1} &= \frac{x-1+1}{x-1} \\ &= 1 + \frac{1}{x-1} \\ &\geq 1 + \frac{1}{(1+\delta)-1} \\ &\geq 1 + M - 1 = M. \end{aligned}$$

Thus we have $\frac{x}{x-1} \geq M$. Since $M$ is arbitrary large, we have $\lim_{x\to 1+} \frac{x}{x-1} = +\infty$, as desired.

**Question 3***. Do Q1, Q2 but for $\lim_{x\to 1-} \frac{x}{x-1} = -\infty$.

**Solution:**

Computation Rule: Suppose $\lim_{x\to x_0} f(x) = c > 0$, and there exists $\delta > 0$ such that $g(x) < 0$ for $x \in (x_0 - \delta, x_0)$. If $\lim_{x\to x_0-} g(x) = 0$, then

$$\lim_{x\to x_0-} \frac{f(x)}{g(x)} = -\infty.$$

*please kindly send an email to cyma@math.cuhk.edu.hk if you have any question.

For any $M < -2$, take $\delta = \frac{1}{-M+1}$. For any $x \in (1-\delta, 1)$,

$$\begin{aligned}\frac{x}{x-1} =& \frac{x-1+1}{x-1} \\ =& 1 - \frac{1}{1-x} \\ \leq& 1 - \frac{1}{\delta} \\ \leq& 1 - (-M+1) = M.\end{aligned}$$

Thus we have $\lim\limits_{x\to 1-} \frac{x}{x-1} = -\infty$.

**Question 5**. Evaluate the following limits, or show that they do not exist.

$$(b) \lim_{x\to 1} \frac{x}{x-1} \quad (x \neq 1), \quad (e) \lim_{x\to 0} (\sqrt{x+1})/x \quad (x > -1).$$

**Solution:**

(b) It follows from $Q1$ and $Q3$ that

$$\lim_{x\to 1+} \frac{x}{x-1} = +\infty, \qquad \lim_{x\to 1-} \frac{x}{x-1} = -\infty.$$

Therefore, the limit $\lim_{x\to 1} \frac{x}{x-1}$ does not exist.

(e) If $x > 0$, then $\sqrt{x+1} > \sqrt{x}$. It follows that

$$\begin{aligned}\lim_{x\to 0^+} \frac{\sqrt{x+1}}{x} &> \lim_{x\to 0^+} \frac{\sqrt{x}}{x} \\ &= \lim_{x\to 0^+} \frac{1}{\sqrt{x}} \\ &= +\infty.\end{aligned}$$

On the other hand, for $x \in (-\frac{1}{2}, 0)$ we have $\sqrt{x+1} > \sqrt{\frac{1}{2}}$, and hence that

$$\frac{\sqrt{x+1}}{x} < \frac{1}{\sqrt{2}x}.$$

It follows that $\lim\limits_{x\to 0-} \frac{\sqrt{x+1}}{x} = -\infty$, since $\lim\limits_{x\to 0-} \frac{1}{\sqrt{2}x} = -\infty$. Therefore we conclude that $\lim\limits_{x\to 0} \frac{\sqrt{x+1}}{x}$ does not exist.