MATH2050B Mathematical Analysis I

Homework 1 suggested Solution*

Question 8. Provide a bounded (= *bounded below and bounded above*) set X of real numbers such that $\min X$, $\max X$ do not exist.

Solution:

Let $X = \{x \in \mathbb{R} : 0 < x < 1\}$. It is easy to see that X is a bounded set. We claim that $\min X$, $\max X$ do not exist. Suppose on the contrary that $\max X$ exists, denoted by a. Since $x < 1$ for any $x \in X$, we have $a < 1$. It follows that

$$0 < a < \frac{1+a}{2} < 1,$$

which contradicts our assumption. Therefore, $\max X$ does not exist.

By a similar argument, we can see that $\min X$ does not exist.

Question 9. (i) Show that for any $x \in \mathbb{R}$,

$$x, -x \leq |x|,$$

and that $x = |x|$ or $-x = |x|$.

(ii) Let $x, y \in \mathbb{R}$ and $0 < \alpha \in \mathbb{R}$. Show that

$$|x| < \alpha \Longleftrightarrow -\alpha < x < \alpha;$$

$$|x - y| < \alpha \Longleftrightarrow x - \alpha < y < x + \alpha.$$

Solution:

(i) Recall that

$$|x| = \begin{cases} x, & x > 0, \\ 0 & x = 0, \\ -x & x < 0. \end{cases}$$

It follows directly that $|x| = x$, if $x \geq 0$; and $x < -x = |x|$, if $x < 0$. Thus, we conclude that $x, -x \leq |x|$ for any $x \in \mathbb{R}$.

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(ii) By (i), we have $|x| < \alpha$ if and only if both $x, -x < \alpha$. This implies

$$|x| < \alpha \Longleftrightarrow -\alpha < x < \alpha.$$

From above argument, we have

$$\begin{aligned}|x-y| < \alpha &\Longleftrightarrow -\alpha < x-y < \alpha \\ &\Longleftrightarrow x-\alpha < y < x+\alpha.\end{aligned}$$

Question 11. We sometimes write (*the notation suggested by looking at the graphs of* $x \mapsto \max\{f(x), g(x)\}$; $x \mapsto \min\{f(x), g(x)\}$; *for real-valued functions* f, g):

$$a \vee b := \max\{a, b\}; \qquad a \wedge b := \min\{a, b\},$$

for any $a, b \in \mathbb{R}$. Show that, $\forall a, b \in \mathbb{R}$,

$$-(a \vee b) = (-a) \wedge (-b); \qquad -(a \wedge b) = (-a) \vee (-b);$$

$$a \vee b = \frac{a+b+|a-b|}{2}; \qquad a \wedge b = \frac{a+b-|a-b|}{2}.$$

Solution:

Without loss of generality, we assume that $a < b$. Thus, $a \vee b = b$ and $a \wedge b = a$.

It follows by $a < b$ that

$$\frac{a+b+|a-b|}{2} = \frac{a+b+b-a}{2} = b.$$

Therefore, we have $a \vee b = \frac{a+b+|a-b|}{2}$. Similarly,

$$\frac{a+b-|a-b|}{2} = \frac{a+b-(b-a)}{2} = a,$$

yields that $a \wedge b = \frac{a+b-|a-b|}{2}$.

Using these formulas, we see that

$$\begin{aligned}(-a) \wedge (-b) &= \frac{(-a)+(-b)-|(-a)-(-b)|}{2} \\ &= \frac{(-a)+(-b)-|(-a)+b|}{2} \\ &= \frac{(-a)+(-b)-(b-a)}{2} \\ &= -b = -(a \vee b);\end{aligned}$$

In the same manner we can see that

$$
\begin{aligned}
(-a) \vee (-b) &= \frac{(-a) + (-b) + |(-a) - (-b)|}{2} \\
&= \frac{(-a) + (-b) + |(-a) + b|}{2} \\
&= \frac{(-a) + (-b) + (b - a)}{2} \\
&= -a = -(a \wedge b).
\end{aligned}
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