THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2050 (First Term)
Mathematical Analysis I
Homework VII

Questions with * will be marked.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be additive: $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Suppose f is continuous at $x_0 = 0$. Show that there exists $c \in \mathbb{R}$ such that $f(x) = cx$ for all $x \in \mathbb{R}$.

2. Let $f(r) = 0$ for all $r \in \mathbb{Q}$. Suppose f is continuous on $\mathbb{R}$. Show that $f(x) = 0$ for all $x \in \mathbb{R}$.

3. * Let $g : \mathbb{R} \to \mathbb{R}$ be such that

$$g(x) = \begin{cases} 2x & \text{if } x \in \mathbb{Q}; \\ x+3 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

   Find the continuity points of g.

4. Let $f : (0\infty), \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \notin \mathbb{Q}; \\ q & \text{if } x = \frac{p}{q} (\text{in the 'standard' representation of} x). \end{cases}$$

   Show that f is unbounded on any interval (of positive length).

5. * Let $f : A \to \mathbb{R}$, $x_0 \in \mathbb{R}$ non-isolated to A and suppose that $\lim\limits_{x \to x_0} f(x)$ does not exist. Show that there exist $\varepsilon > 0$ and two sequences (x_n), (y_n) in $A \setminus \{x_0\}$ converge to x_0 such that $|f(x_n) - f(y_n)| \geq \varepsilon$ for all n.

   If f is bounded (in the sense that the range of f is a bounded subset of $\mathbb{R}$), show further that there exist two sequences (x'_n) and (y'_n) in $A \setminus \{x_0\}$ converge to x_0 such that $\lim f(x'_n) = \ell' \neq \ell'' = \lim f(y'_n)$.

6. * Consider real numbers $a < b < c$. Let $f : (a, b] \to \mathbb{R}$ and $g : [b, c) \to \mathbb{R}$ be continuous at b, and suppose that $f(b) = g(b)$. Let $h : (a, c) \to \mathbb{R}$ be defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \in (a, b]; \\ g(x) & \text{if } x \in [b, c). \end{cases}$$

   Show that

   (a) h is continuous at b;

   (b) if f, g are uniformly continuous then h is uniformly continuous.