Th 1 (Boundedness + Max-Min Value Theorems). Let $I=[a,b]$ $(-\infty<a\le b<+\infty)$, and $f: I\to\mathbb{R}$ be cts. Then

(i) f is (globally) bounded: $\{f(x) : x\in I\}$ is bounded;

(ii) $\exists\, x_*, x^*\in I$ s.t.

$$f(x_*)\le f(x)\le f(x^*) \quad \forall x\in I, \qquad (*)$$

that is

$$M^* \overset{\text{def}}{=} f(x_*) = \inf\{f(x) : x\in I\} = \min\{f(x) : x\in I\}.$$

and

$$M_* \overset{\text{def}}{=} f(x^*) = \cdots$$

Th 2 (Root Th + Intermediate Value Th) Let $I=[a,b]$ and $f: I\to\mathbb{R}$ be cts as in Th 1; let $k\in\mathbb{R}$. Then the following statements are valid:

(i) If $f(a)f(b)<0$ then $\exists\, c\in(a,b)$ s.t. $f(c)=0$

(so if $f(a)f(b)\le 0$ then $\exists\, c\in[a,b]$ s.t. $f(c)=0$)

(ii) If $f(a)<k<f(b)$ (or $f(a)>k>f(b)$) then $\exists\, c\in(a,b)$ such that $f(c)=k$ (so if $f(a)\le k\le f(b)$ or $f(a)\ge k\ge f(b)$ then $\exists\, c\in[a,b]$ s.t. $f(c)=k$).

Cor 1. Under the same assumption as in Th 1 & 2: f is a cts function on $[a,b]$. Then $\{f(x) : x\in[a,b]\}=[M_*, M^*]$ where M_*, M^* are defined in Th 1.

Proof. Let $k\in[M_*, M^*]$. Then $f(x_*)\le k\le f(x^*)$, where x_*, x^* are as in Th 1. By (ii) (applied to the interval I^* with end-points x_*, x^* (so $I^*\subseteq[a,b]$)), $\exists\, c\in I^*\subseteq[a,b]$ s.t. $k=f(c)$ showing that

$$\{f(x) : x\in[a,b]\}\supseteq[M_*, M^*].$$

The converse inclusion also holds by $(*)$.

Cor 2. Let I be <u>any</u> nonempty interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be cts. Then $f(I) := \{f(x) : x \in I\}$ is also an interval (as $\{f(x) : x \in I\}$ is order convex by Th 2(ii) : pl provide more detail as an exercise).

§ 5.4. Uniform Continuity.

Recall the definitions (continuity vs <u>unif.</u> continuity)

Ex 1. x^2 is continuous on $(-\infty, \infty)$ but <u>not</u> unif cts.

Ex 2. $\frac{1}{x} = f(x)$ is continuous on $(0, 1]$ but not unif cts.

(Let $\varepsilon := 1/2$, and let $\delta > 0$. Take $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \delta$, and let $x' = \frac{1}{n}$ and $x'' = \frac{1}{2n}$. Then

$$\|x' - x''\| = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n} < \delta$$

but

$$|f(x') - f(x'')| = 2n - n = n > \varepsilon .$$

Therefore $f\left(x \mapsto \frac{1}{x}\right)$ is not cts on $(0, 1]$)

Th (Uniform Continuity Th). Let $I = [a, b]$ be a nonempty bounded closed interval $(a \leq b)$, and let $f : [a, b] \to \mathbb{R}$. Then $\circlearrowright$:

(i) f is continuous

(ii) f is uniformly cts

Proof. May assume that $a < b$. Suffices to show that (i) $\Rightarrow$ (ii). So suppose f is cts on $[a,b]$. Suppose (ii) is not true; we seek a contradiction. By the negation of (ii), $\exists\ \varepsilon > 0$ such that each $\delta > 0$ fails the following property

$$|f(x) - f(x')| < \varepsilon \text{ whenever } |x - x'| < \delta \text{ with } x, x' \in [a,b]$$

Then, $\forall\ n \in \mathbb{N}$, $\exists\ x_n, x_n' \in [a,b]$ with $|x_n - x_n'| < \frac{1}{n}$ but $|f(x_n) - f(x_n')| \geq \varepsilon$. To this for all $n \in \mathbb{N}$, we have sequences $(x_n), (x_n')$ in $[a,b]$ satisfying

$$|x_n - x_n'| < \frac{1}{n} \quad \forall\ n \in \mathbb{N} \tag{1}$$

and

$$|f(x_n) - f(x_n')| \geq \varepsilon \quad \forall\ n \in \mathbb{N}. \tag{2}$$

By B-W theo together with the order-preserving for sequential limits, $\exists$ a subseq (x_{n_k}) of (x_n) convergent to some $\bar{x} \in [a,b]$. By (i), f is cts at $\bar{x}$ and it follows from the sequential criterion for continuity that

$$\lim_k f(x_{n_k}) = f(\bar{x}). \tag{3}$$

On the other hand, by (1) and the Squeeze Principle that $\lim_n \left(f(x_n) - f(x_n')\right) = 0$ and so

$$\lim_k f(x_{n_k'}) = \lim_k \left(f(x_{n_k'}) - f(x_{n_k}) + f(x_{n_k}) \right)$$

$$= 0 + \lim_k f(x_{n_k}) = f(\bar{x}),$$

thanks to (3) and computation rules. Thus

$$\lim_k \left(f(x_{n_k}) - f(x_{n_k'})\right) = f(\bar{x}) - f(\bar{x}) = 0,$$

contradicting to (2).

Further examples (relating p135-136).

<u>Q14</u>. Let $f: D \to \mathbb{R}$ be cts at $x_0 \in D$ s.t. $f(x_0) < \alpha$. Then $\exists\, \delta > 0$ s.t $f(x) < \alpha \ \forall x \in V_\delta(x_0) \cap D$.

<u>Sol</u>. Let $\varepsilon := \alpha - f(x_0)\ (>0)$. Then, by continuity, $\exists\, \delta > 0$ s.t

$$f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon, \quad \forall x \in V_\delta(x_0) \cap D.$$

Since $f(x_0) + \varepsilon = \alpha$, this implies in particular that

$$f(x) < f(x_0) + \varepsilon = \alpha, \quad \forall x \in V_\delta(x_0) \cap D.$$

Remark. Similar results for $\beta < f(x_0)$ or for

$$\beta < f(x_0) < \alpha.$$

<u>Q17</u> Let $f: [0,1] \to \mathbb{R}$ be cts such that $f(x) \in \mathbb{Q}\ \forall x \in [0,1]$. Then f must be a constant function: $f(x_1) = f(x_2)$ $\forall x_1, x_2 \in [a,b]$.

<u>Sol</u>. Suppose $\exists\, x_1, x_2 \in [0,1]$ s.t $f(x_1) < f(x_2)$. By density of $\mathbb{R} \setminus \mathbb{Q}$, $\exists\, k \in \mathbb{R} \setminus \mathbb{Q}$ s.t. $f(x_1) < k < f(x_2)$. Then, by

Th 2 (ii), $\exists\ x_0$ lying between x_1 and x_2 (so $x_0 \in [0,1]$) such that $f(x_0) = k$, contradicting to the given assumption.

Q12. Let $f = f_1 \vee f_2$ where $f_1, f_2 : [a,b] \to \mathbb{R}$ be cts such that

$$f_1(a) < f_2(a) \quad \text{and} \quad f_1(b) > f_2(b) \qquad (1)$$

(e.g. f_1, f_2 are respective x^2, $\cos x$ on $[0, \pi/2]$). Show that $\exists$ a global minimum-point x_* ~~of~~ for f and that this x_* satisfies the equation $f_1(x) = f_2(x)$ if f_1, f_2 are strictly monotone on $[a,b]$ (i.e. $f_1(x) < f_1(x')$ and $f_2(x) > f_2(x')$ whenever $x < x'$ with $x, x' \in [a,b]$).

Hint of Solution. The existence of x_* is already done in Th 1 (as f is cts on $[a,b]$). By assumptions $\exists\ a' > a$ and $b' < b$ such that $f = f_2$ strictly ~~in~~decreasing on (a,a') and $f = f_1$ strictly increasing on (b', b); hence ~~it~~ the minimum-pt $x_* \neq a$ & $x_* \neq b$. Show further that each of the cases 1) $f_1(x_*) < f_2(x_*)$ and 2) $f_1(x_*) > f_2(x_*)$ is not possible. e.g., In case 1), $\exists\ \delta > 0$ such that $V_\delta(x_*) \subset (a,b)$ and $f_1(\cdot) < f_2(\cdot)$ on $V_\delta(x_*)$,