Global Solutions of Nematic Liquid Crystal Flow in Dimension Two

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ABSTRACT: In this article we are concerned with a simplified Ericksen-Leslie system on $\mathbb{R}^2$, whose bounded domain case was considered by Lin-Lin-Wang in [20]. With a study of its vorticity-stream formulation, we establish a global existence result of weak solutions when initial orientation has finite energy and initial vorticity function lies in $\mathrm{L}^1(\mathbb{R}^2)$.

I. INTRODUCTION

I.1. BACKGROUND AND MOTIVATION Ericksen-Leslie system is a hydrodynamic system modeling the flow of nematic liquid crystals. Proposed in [7], [19] and references therein, it is a continuum theory without molecular details of a liquid crystal material. Recently some research works have been devoted to studying the relationships between the theory of Ericksen-Leslie and two other favorable theories (Doi-Onsager theory and Landau-de Gennes theory) for nematic liquid crystals. In [29] the Doi-Onsager theory (see [6] and [25]) is connected with the Ericksen-Leslie theory by taking the Deborah number to zero. As a hydrodynamic Landau-de Gennes model, the Beris-Edwards system (see [4]) was studied by the authors in [26]-[28]. Particularly in [28], a Hilbert expansion was obtained for solutions of the Beris-Edwards system with which a well prepared initial data is supplied. When elastic constants are small, their work rigorously shows that the Ericksen-Leslie system serves as the limit of the Beris-Edwards system before the first singular time. For the static theory of liquid crystals, readers should be referred to [1] and [24] for important connections and differences between the Landau-de Gennes theory and the Oseen-Frank theory. As far as the Ericksen-Leslie system is concerned, many research works have been established on its well-posedness. In 2-D case, the existence of global weak solution for a simplified Ericksen-Leslie equation has been obtained by the authors in [20], where the domain is supposed to be bounded and smooth. The associated uniqueness problem was later studied by Lin-Wang in [21]. In [14] the author considered the same simplified Ericksen-Leslie equation but on the whole space $\mathbb{R}^2$. When the spatial domain is $\mathbb{R}^2$ and the model is not restricted to the simplified one studied in [20], the global existence of weak solutions for the Ericksen-Leslie system with general Oseen-Frank energy are also well studied (see [15]-[16]). Amongst all the works in 2-D, global weak solutions have finite energy and are smooth except possibly at finitely many singularities. Compared with the 2-D case, our knowledge on the 3-D Ericksen-Leslie system is limited. In [30] the authors established the local well-posedness of the general Ericksen-Leslie system. For the sake of describing its maximal existence time interval, a blow-up criterion (same as the one in [17]) is given. With this criterion, the authors proceed to prove the global existence of the general Ericksen-Leslie system under the assumption that initial data is small in some Sobolev spaces. The spatial domain in [30] is $\mathbb{R}^3$. For the bounded smooth domain in $\mathbb{R}^3$, the authors in [23] also established a global existence result for weak solution of simplified Ericksen-Leslie equation. Different from [30], the consequence in [23] does not rely on the smallness of initial data in Sobolev spaces. Instead Lin-Wang made a geometrically small assumption in [23] for their initial data. More precisely by supposing that initial macroscopic orientation takes its image on the upper hemisphere, the simplified Ericksen-Leslie equation studied in [20] admits a global weak solution on any bounded smooth domain in $\mathbb{R}^3$, where initial data is only required to be in the natural energy space. For more detailed mathematical studies of nematic liquid crystals, readers are referred to [22].

Without macroscopic orientation, the Ericksen-Leslie system is reduced to the pure Navier-Stokes equation. It is well-known that the Navier-Stokes equation admits a vorticity-stream formulation (see [5]). For the 2-D viscous fluid, taking curl of the Navier-Stokes equation leads to the following vorticity equation:

$$\partial_t \omega + v \cdot \nabla \omega = \Delta \omega.$$

Here v is the velocity of fluid. $\omega = \mathrm{curl}\, v$ is its vorticity. In [2] and the references therein the global existence of the above vorticity equation is studied in $\mathbb{R}^2$, where the velocity v is recovered by the Biot-Savart law. Initial vorticity is assumed to be in the L^1-space. In [12] (see also [3]), the regularity of initial data is slightly weakened. The global existence of the vorticity equation in 2-D is shown to hold with given initial data in the Radon measure space on $\mathbb{R}^2$. Besides the global existence result of the vorticity equation, the stability problem associated with the Navier-Stokes equation in 2-D is also considered with the use of the above vorticity equation (see e.g. [9] and [11]-[12]). In [12] the authors studied the long-time behavior of the vorticity of the 2-D Navier-Stokes equation. With a smallness assumption on the Reynolds number of initial vorticity, it is shown that solutions of the vorticity equation approach to the so-called Oseen's vortex as $t \to \infty$. The convergence is algebraic in t. Still in [12], this result was further applied to study the stability of Burger's vortex for 3-D Navier-Stokes equation. Later in [9] and [11], the authors considered the long-time behavior of vorticity and its stability for the 2-D Navier-Stokes equation from the point of view of dynamical system. Finally in [11] the authors dropped the smallness assumption used in [12] for the Reynolds number of initial vorticity. A global stability result is obtained by LaSalle's invariance principle and the theory of Lyapunov. Some stability results on the 3-D Navier-Stokes equation can be read from [10].

I.2. VORTICITY EQUATION OF ERICKSEN-LESLIE SYSTEM In this article we are concerned with the simplified hydrodynamic system for nematic liquid crystals studied by Lin-Lin-Wang in [20]. The spatial domain is supposed to be $\mathbb{R}^2$. With all parameters in the system normalized to be 1, the equation is written as follows:

$$\begin{cases} \partial_t \phi + v \cdot \nabla \phi - \Delta \phi = |\nabla \phi|^2 \phi\,, & \text{in } \mathbb{R}^2 \times (0, \infty); \\ \partial_t v + v \cdot \nabla v - \Delta v = -\nabla p - \nabla \cdot \big(\nabla \phi \odot \nabla \phi \big)\,, & \text{in } \mathbb{R}^2 \times (0, \infty); \\ \nabla \cdot v = 0, & \text{in } \mathbb{R}^2 \times (0, \infty). \end{cases} \tag{1.1}$$

In (1.1) ϕ is an $\mathbb{S}^2$-valued macroscopic orientation of a nematic liquid crystal. $v : \mathbb{R}^2 \times (0, +\infty) \to \mathbb{R}^2$ represents the velocity of fluid. p is the pressure function. $\nabla\phi \odot \nabla\phi$ denotes the 2×2 matrix whose entry on the i-th row and j-th column is given by $\partial_i \phi \cdot \partial_j \phi$. As one can see, system (1.1) is a coupled system between the non-homogeneous incompressible Navier-Stokes equation and the transported flow of harmonic maps. Since early studies of fluid dynamics, problems associated with "singular objects" have been intriguing a lot of attentions from both mathematicians and physicists. These singular objects include point vortices and vortex filaments in fluid dynamics, which are related to vortex phenomena of a fluid. Usually a system with such singular objects might not have a finite kinetic energy, or equivalently square integrable velocity. Explicit examples can be given by the so-called Oseen vortices (see [11]). For some rigorous proof one may refer to [5], where the authors show that for an incompressible velocity recovered by the Biot-Savart law (vorticity has compact support in $\mathbb{R}^2$), it has finite kinetic energy if and only if the total vorticity equals to 0. Thus to study some vortex phenomenon associated with (1.1), it is more convenient to consider the equation of vorticity instead of velocity. In light of the above arguments, now we take curl on both sides of the second equation in (1.1). Still using Biot-Savart law to recover velocity from vorticity, we can rewrite (1.1) in terms of the vorticity of v. That is the system:

$$\begin{cases} \partial_t \phi + v \cdot \nabla \phi - \Delta \phi = |\nabla \phi|^2 \phi\,, & \text{in } \mathbb{R}^2 \times (0, \infty); \\ \qquad v = \mathrm{K} * \omega\,, & \text{in } \mathbb{R}^2 \times (0, \infty); \\ \partial_t \omega + v \cdot \nabla \omega - \Delta \omega = -\nabla \times \nabla \cdot \big(\nabla \phi \odot \nabla \phi \big)\,, & \text{in } \mathbb{R}^2 \times (0, \infty). \end{cases} \tag{1.2}$$

In (1.2) $*$ denotes the standard convolution operator on $\mathbb{R}^2$. For all $x = (x_1, x_2) \in \mathbb{R}^2$, $\mathrm{K}(x)$ is the Biot-Savart kernel given by

$$\mathrm{K}(x) = \frac{1}{2\pi}\,\frac{x^\perp}{|x|^2},$$

where $x^\perp = (-x_2\,,\,x_1\,)$. Note that $\nabla\cdot \mathrm{K} = 0$ implies the incompressibility condition: $\nabla\cdot v = 0$. In the remaining of the article, (1.2) is referred as the vorticity equation of the Ericksen-Leslie system (1.1).

I.3. MAIN RESULTS AND ORGANIZATION OF THE ARTICLE Our first theorem is concerned with the local existence of classical solutions to (1.2). Before we state the result, some notions should be given. First of all we introduce some functional spaces in Definition 1.1, which will be used to control velocity field v recovered by the Biot-Savart law. Since $v = \mathrm{K} * \omega$ for some vorticity function ω, the decay of ω at spatial infinity plays important roles in estimating the Hölder norm and the kinetic energy of v. However the standard Hölder norms and L^p norms are not strong enough to control the decay of ω at spatial infinity. Therefore we introduce the following $\mathrm{C}^{*,k}_\beta\big[\,I\,\big]$ and $\mathrm{C}^*_\beta\left(\mathbb{R}^2\right)$ spaces, in which functions decay exponentially at spatial infinity.

Definition 1.1. *Suppose that f takes value on some Euclidean space. Given a positive constant β and a finite time interval I, we say $f \in \mathrm{C}^*_\beta\big[\,I\,\big]$ if f is continuous on $\mathbb{R}^2 \times I$ and satisfies*

$$|||\, f \,|||_{\beta\,;\,I} := \sup\left\{\,\big|\,f(x,t)\,\big|\, e^{|\,x\,|/\beta} : (\,x\,,\,t\,) \in \mathbb{R}^2 \times I\,\right\} < \infty.$$

$|||\cdot|||_{\beta\,;\,I}$ defines a norm on the space $\mathrm{C}^*_\beta\big[\,I\,\big]$. Equipped with this norm, $\mathrm{C}^*_\beta\big[\,I\,\big]$ is a Banach space. Given a $k \in \mathbb{N}$, we denote by $\mathrm{C}^{*,k}_\beta\big[\,I\,\big]$ the function space so that for all $f \in \mathrm{C}^{*,k}_\beta\big[\,I\,\big]$, it satisfies $\nabla^i f \in \mathrm{C}^*_\beta\big[\,I\,\big]$. Here the index i runs from 0 to k. $\mathrm{C}^{*,k}_\beta\big[\,I\,\big]$ is also a Banach space with norm given by

$$|||\, f \,|||_{k\,;\,\beta\,;\,I} := \sum_{i=0}^{k}\, |||\, \nabla^i f \,|||_{\beta\,;\,I}.$$

Similarly we define $\mathrm{C}^*_\beta\left(\mathbb{R}^2\right)$ to be the space so that for all $f \in \mathrm{C}^*_\beta\left(\mathbb{R}^2\right)$, it holds

$$|||\, f \,|||_{\beta} := \sup\left\{\,\big|\,f(x)\,\big|\, e^{|\,x\,|/\beta} : x \in \mathbb{R}^2\,\right\} < \infty.$$

Equipped with this norm, $\mathrm{C}^*_\beta\left(\mathbb{R}^2\right)$ is a Banach space. Given a $k \in \mathbb{N}$, $\mathrm{C}^{*,k}_\beta\left(\mathbb{R}^2\right)$ denotes the function space so that for all $f \in \mathrm{C}^{*,k}_\beta\left(\mathbb{R}^2\right)$, it satisfies $\nabla^i f \in \mathrm{C}^*_\beta\left(\mathbb{R}^2\right)$. Here i runs from 0 to k. The space $\mathrm{C}^{*,k}_\beta\left(\mathbb{R}^2\right)$ is also a Banach space with norm given by

$$|||\, f \,|||_{k\,;\,\beta} := \sum_{i=0}^{k}\, |||\, \nabla^i f \,|||_{\beta}.$$

In Sect.II we are concerned with some important properties associated with the functional spaces given in Definition 1.1. With these properties, the following theorem is shown in Sect.III by a fixed-point argument. Notice that in Theorem 1.2 below, we call (ϕ,ω) a classic solution of (1.2) on $\mathbb{R}^2 \times \big[0,T\big]$ if on this domain $\big(\partial_t\phi, \partial_t\omega\big)$, $\big(\nabla^i\phi, \nabla^i\omega\big)$ $(i = 0,1,2)$ are continuous and satisfy (1.2) in a pointwise sense.

Theorem 1.2. *Suppose that $\omega_0 \in \mathrm{C}^{*,2}_2\left(\mathbb{R}^2\right)$ and ϕ_0 is an $\mathbb{S}^2$-valued function with $\phi_0 - e \in \mathrm{C}^{*,4}_1\left(\mathbb{R}^2\right)$. Then there exists a $T_* > 0$ such that (1.2) admits a classic solution on $\mathbb{R}^2 \times [0,T_*]$ with the given initial data (ϕ_0,ω_0). If we denote by (ϕ,ω) the classic solution, then we also have*

$$\big(\phi - e,\,\omega\big) \in \mathrm{C}^{*,4}_1\,[0,T_*] \times \mathrm{C}^{*,2}_2\,[0,T_*].$$

Here $e \in \mathbb{S}^2$ is a constant unit vector in $\mathbb{R}^3$.

Our next theorem is about the local existence of solutions for (1.2) with initial data $(\phi_0, \omega_0) \in \mathrm{H}^1_e(\mathbb{R}^2; \mathbb{S}^2) \times \mathrm{L}^1(\mathbb{R}^2)$. Here for a given $e \in \mathbb{S}^2$, $\mathrm{H}^1_e(\mathbb{R}^2; \mathbb{S}^2)$ denotes the space given below:

$$\mathrm{H}^1_e(\mathbb{R}^2; \mathbb{S}^2) := \Big\{ \phi : \phi(x) \in \mathbb{S}^2 \text{ for almost all } x \in \mathbb{R}^2 \text{ and } \phi - e \in \mathrm{H}^1(\mathbb{R}^2) \Big\}.$$

Approximating $(\phi_0 - e, \omega_0)$ by a sequence of smooth pairs with compact support, we can find a sequence of solutions of (1.2) whose initial data equal to the smooth pairs. In Sect. IV, we show that these solutions exist in a uniform time interval. Thus by appropriate compactness arguments, we can show

Theorem 1.3. *Suppose that (ϕ_0, ω_0) is an initial data in $\mathrm{H}^1_e(\mathbb{R}^2; \mathbb{S}^2) \times \mathrm{L}^1(\mathbb{R}^2)$. Then there exists a $T_* > 0$ and a smooth solution, denoted by (ϕ, ω), of (1.2) on $(0, T_*)$ so that the following properties hold:*

(i). *As $t \downarrow 0$, we have*

$$\big(\phi(\cdot, t) - e, \omega(\cdot, t)\big) \longrightarrow \big(\phi_0 - e, \omega_0\big), \qquad \textit{strongly in } \mathrm{H}^1(\mathbb{R}^2) \times \mathrm{L}^1(\mathbb{R}^2).$$

Let $\big(\mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{L}^p(\mathbb{R}^2)\big)^*$ be the dual space of $\mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{L}^p(\mathbb{R}^2)$. Then as $t \downarrow 0$, the velocity $v = \mathrm{K} * \omega$ satisfies

$$v(\cdot, t) \longrightarrow v_0 = \mathrm{K} * \omega_0, \qquad \textit{strongly in } \big(\mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{L}^p(\mathbb{R}^2)\big)^*, \textit{ for all } p > 2.$$

Here we equip the space $\mathrm{L}^1(\mathbb{R}^2) \cap \mathrm{L}^p(\mathbb{R}^2)$ with the norm defined by $\|\cdot\|_1 + \|\cdot\|_p$. Moreover we also have

$$\big(\phi - e, \omega\big) \in \mathrm{L}^\infty\big([0, T_*]; \mathrm{H}^1(\mathbb{R}^2)\big) \times \mathrm{L}^\infty\big([0, T_*]; \mathrm{L}^1(\mathbb{R}^2)\big).$$

(ii). *Fixing a $\tau \in (0, T_*)$ and denoting by $\bar{\omega}$ the unique mild solution (see Chapter 4 of [3]) of the following initial value problem:*

$$\begin{cases} \partial_t \bar{\omega} - \Delta \bar{\omega} + \bar{v} \cdot \nabla \bar{\omega} = 0, & \textit{on } \mathbb{R}^2 \times (\tau, \infty); \\ \bar{\omega}(\cdot, \tau) = \omega(\cdot, \tau); \quad \bar{v} = \mathrm{K} * \bar{\omega}, & \end{cases} \tag{1.3}$$

then we can decompose the velocity field v into the sum

$$v = \bar{v} + v^*, \qquad \textit{on } \mathbb{R}^2 \times [\tau, T_*]. \tag{1.4}$$

The velocity field v^* lies in the space $\mathrm{L}^\infty\big([\tau, T_*]; \mathrm{L}^2(\mathbb{R}^2)\big) \cap \mathrm{L}^2\big([\tau, T_*]; \mathrm{H}^1(\mathbb{R}^2)\big)$ and satisfies the global energy inequality given below:

$$\int_{\mathbb{R}^2 \times \{t_2\}} |v^*|^2 + |\nabla \phi|^2 \quad + \quad \int_{t_1}^{t_2} \int_{\mathbb{R}^2} |\nabla v^*|^2 + \big|\Delta \phi + |\nabla \phi|^2 \phi\big|^2$$
$$\leqslant \exp\Big\{ \mathrm{c} \int_{t_1}^{t_2} \|\nabla \bar{v}\|_\infty \Big\} \int_{\mathbb{R}^2 \times \{t_1\}} |v^*|^2 + |\nabla \phi|^2. \tag{1.5}$$

Here $c > 0$ is an universal constant. t_1 and t_2 satisfy $\tau \leqslant t_1 < t_2 \leqslant T_*$. Moreover as $t \downarrow \tau$, $v^*(\cdot, t)$ converges to 0 strongly in L^2.

(iii). *If $\omega_0 \in \mathrm{L}^1 \cap \mathrm{L}^p$ for some $p > 1$, then τ in part (ii) can take value 0. The decomposition of the velocity field v in (1.4) holds on $\mathbb{R}^2 \times [0, T_*]$.*

We are also concerned about the global weak solutions of (1.1). Notice the decomposition of v in (1.4). $\bar{v}$ already exists on the time interval (τ, ∞). Therefore to extend v globally in time, we just need extend v^* to $\mathbb{R}^2 \times (\tau, \infty)$. In light of the global energy inequality (1.5), such extension of v^* is expected . As a consequence, we have

Theorem 1.4. *Given* $(\phi_0, \omega_0) \in \mathrm{H}^1_e\big(\mathbb{R}^2; \mathbb{S}^2\big) \times \mathrm{L}^1\big(\mathbb{R}^2\big)$, *there exists a global weak solution of (1.1) in the sense given as follows:*

(i). *For some* $T_* > 0$, (ϕ, v) *is a smooth solution of (1.1) on* $\mathbb{R}^2 \times (0, T_*)$. *Moreover parts (i)-(ii) in Theorem 1.3 hold for* (ϕ, v, ω), *where* ω *is the vorticity of* v;

(ii). *Let* $(\bar{\omega}, \bar{v})$ *be the same as in part (ii) of Theorem 1.3. Then on* $\mathbb{R}^2 \times [\tau, \infty)$, v *can be decomposed into the sum* $v = \bar{v} + v^*$. (ϕ, v^*) *satisfies the global energy inequality (1.5) for all* t_1 *and* t_2 *satisfying* $\tau \leqslant t_1 < t_2 < \infty$. *Moreover* (ϕ, v^*) *is a global weak solution of the following system:*

$$\begin{cases} \partial_t \phi + v^* \cdot \nabla \phi - \Delta \phi = -\bar{v} \cdot \nabla \phi + |\nabla \phi|^2 \phi\,, & \text{on } \mathbb{R}^2 \times (\tau, \infty); \\ \partial_t v^* + v^* \cdot \nabla v^* - \Delta v^* = -v^* \cdot \nabla \bar{v} - \bar{v} \cdot \nabla v^* - \nabla p^* - \nabla \cdot \big(\nabla \phi \odot \nabla \phi\big), & \text{on } \mathbb{R}^2 \times \big(\tau, \infty\big); \\ \nabla \cdot v^* = 0, \end{cases}$$

together with the initial condition:

$$\big(\phi, v^*\big)\big|_{t=\tau} = \big(\phi(\cdot, \tau), 0\big).$$

More precisely it holds

$$\begin{aligned} -\int_\tau^T \int_{\mathbb{R}^2} \big\langle \phi - e, \eta' \psi \big\rangle \;&+\; \int_\tau^T \int_{\mathbb{R}^2} \big\langle v^* \cdot \nabla \phi, \eta \psi \big\rangle \;+\; \int_\tau^T \int_{\mathbb{R}^2} \eta \nabla \psi : \nabla \phi \\ &= \eta(\tau) \int_{\mathbb{R}^2} \big\langle \phi(\cdot, \tau) - e, \psi \big\rangle - \int_\tau^T \int_{\mathbb{R}^2} \big\langle \bar{v} \cdot \nabla \phi, \eta \psi \big\rangle + \int_\tau^T \int_{\mathbb{R}^2} \big|\nabla \phi\big|^2 \big\langle \phi, \eta \psi \big\rangle; \end{aligned}$$

and

$$\begin{aligned} -\int_\tau^T \int_{\mathbb{R}^2} \big\langle v^*, \eta' \varphi \big\rangle \;&+\; \int_\tau^T \int_{\mathbb{R}^2} \big\langle v^* \cdot \nabla v^*, \eta \varphi \big\rangle + \int_\tau^T \int_{\mathbb{R}^2} \eta \nabla v^* : \nabla \varphi \\ &= -\int_\tau^T \int_{\mathbb{R}^2} \big\langle v^* \cdot \nabla \bar{v}, \eta \varphi \big\rangle - \int_\tau^T \int_{\mathbb{R}^2} \big\langle \bar{v} \cdot \nabla v^*, \eta \varphi \big\rangle + \int_\tau^T \int_{\mathbb{R}^2} \eta \nabla \phi \odot \nabla \phi : \nabla \varphi, \end{aligned}$$

for all $T \in [\tau, \infty]$, $\psi \in \mathrm{H}^1\big(\mathbb{R}^2; \mathbb{R}^3\big)$, $\varphi \in \mathrm{H}^1_{\mathrm{div}}\big(\mathbb{R}^2; \mathbb{R}^2\big)$ *and* $\eta \in \mathrm{C}^\infty[\tau, T]$ *with* $\eta(T) = 0$. *Here*

$$\mathrm{H}^1_{\mathrm{div}}\big(\mathbb{R}^2; \mathbb{R}^2\big) = \text{ closure of } \mathrm{C}^\infty_c\big(\mathbb{R}^2; \mathbb{R}^2\big) \cap \big\{v : \operatorname{div} v = 0\big\} \text{ in } \mathrm{H}^1\big(\mathbb{R}^2; \mathbb{R}^2\big).$$

I.4. NOTATIONS In this article we use L^p, $W^{k,p}$ and $\mathrm{C}^{k,\alpha}$ to denote the standard L^p-space, $W^{k,p}$-Sobolev spaces and $\mathrm{C}^{k,\alpha}$-spaces on $\mathbb{R}^2$. The corresponding norms are denoted by $\|\cdot\|_p$, $\|\cdot\|_{k,p}$ and $\|\cdot\|_{\mathrm{C}^{k,\alpha}}$, respectively. For the Hölder space C^α, we also use $[\cdot]_\alpha$ to denote its semi-Hölder norm. If $p = 2$, then we use H^k to denote the Sobolev spaces $W^{k,2}$. On the space-time $\mathbb{R}^2 \times \mathrm{I}$, where I is an arbitrary time interval, we say a function is $\mathrm{C}^{\alpha/2,\alpha}$ if it is $\mathrm{C}^{\alpha/2}$-Hölder continuous with respect to the time variable and C^α-Hölder continuous with respect to the space variables. Some times we also use $|\cdot|_{0;\mathrm{I}}$ to denote the L^∞-norm of a continuous function on $\mathbb{R}^2 \times \mathrm{I}$. Letting X be a functional space on $\mathbb{R}^2$ with norm $\|\cdot\|_X$, usually we denote by $\mathrm{L}^p\big(\mathrm{I}; X\big)$ the space so that for all $f \in \mathrm{L}^p\big(\mathrm{I}; X\big)$, $f(\cdot, t)$ lies in X for almost every $t \in \mathrm{I}$ and $\|f(\cdot, t)\|_X$ is L^p-integrable on I. If $g(\cdot, t)$ is a continuous mapping from I to X with topology on X induced by $\|\cdot\|_X$, then we call $g \in \mathrm{C}[\mathrm{I}; X]$. Moreover in this article $A \lesssim B$ means that there is a universal constant c so that $A \leqslant cB$. If we want to emphasize the dependence of c on parameters a and b, then we use the notation $A \lesssim_{a,b} B$.

II. PRELIMINARY RESULTS

This section is devoted to studying some basic properties associated with functions in $\mathrm{C}^{*,k}_{\beta}[I]$ and $\mathrm{C}^{*,k}_{\beta}(\mathbb{R}^2)$ (see Definition 1.1). k is a non-negative integer. When $k = 0$, the spaces $\mathrm{C}^{*,0}_{\beta}[I]$ and $\mathrm{C}^{*,0}_{\beta}(\mathbb{R}^2)$ are coincident with $\mathrm{C}^{*}_{\beta}[I]$ and $\mathrm{C}^{*}_{\beta}(\mathbb{R}^2)$, respectively. The first lemma is about solution of a nonhomogeneous linear heat equation with nonhomogeneous term in $\mathrm{C}^{*}_{\beta}[0,T]$. Throughout the article we use G to denote the standard heat kernel in $\mathbb{R}^2$.

Lemma 2.1. *Suppose that β and T are two positive constants. α and θ are two constants in $(0,1)$. Given g a function in $\mathrm{L}^{\infty}\big([0,T];\mathrm{C}^{\alpha}(\mathbb{R}^2)\big)\cap \mathrm{C}^{*}_{\beta}[0,T]$, we define*

$$\Phi[g](x,t) = \int_0^t\int_{\mathbb{R}^2} \mathrm{G}(x-z,t-s)\,g(z,s)\,\mathrm{d}z\,\mathrm{d}s, \qquad \forall\,(x,t)\in\mathbb{R}^2\times[0,T]. \tag{2.1}$$

Then $\Phi[g]$ is a solution of the following nonhomogenous Cauchy problem:

$$\begin{cases} \partial_t f - \Delta f = g, & \text{in } \mathbb{R}^2\times[0,T];\\ f\equiv 0\,, & \text{at } t=0. \end{cases} \tag{2.2}$$

Moreover $\Phi[g]$ satisfies the estimate given below:

$$|||\,\Phi[g]\,|||_{\beta;\,[0,T]} + T^{1/2}\,|||\,\nabla\Phi[g]\,|||_{\beta;\,[0,T]} \;\lesssim\; |||\,g\,|||_{\beta;\,[0,T]}\,T\,e^{2T/\beta^2}. \tag{2.3}$$

The second order derivatives of $\Phi[g]$ can be estimated by

$$|||\,\nabla^2\Phi[g]\,|||_{\beta/\theta;\,[0,T]} \;\lesssim_{\theta,\alpha}\; \max_{t\in[0,T]}\big[g(\cdot,t)\big]_{\alpha}^{1-\theta}\,|||\,g\,|||^{\theta}_{\beta;\,[0,T]}\,T^{(1-\theta)\alpha/2}\,e^{2T\theta^2/\beta^2}. \tag{2.4}$$

Proof. By Theorem 12 in Chapter 1 of [8], $\Phi[g]$ is a solution of (2.2). Moreover $\Phi[g]$, $\partial_t\Phi[g]$ and $\nabla^i\Phi[g]$ $(i=1,2)$ are continuous on $\mathbb{R}^2\times[0,T]$. Thus we are left to show (2.3)-(2.4). Let f denote the function $\Phi[g]$. By (2.1) and the norm $|||\cdot|||_{\beta;\,[0,T]}$ given in Definition 1.1, $f(x,t)$ can be estimated as follows:

$$|\,f(x,t)\,| \;\leqslant\; |||\,g\,|||_{\beta;[0,T]}\int_0^t\int_{\mathbb{R}^2}\frac{1}{4\pi(t-s)}\,e^{-|x-z|^2/4(t-s)}\,e^{-|z|/\beta}\,\mathrm{d}z\,\mathrm{d}s.$$

Applying the change of variable $\xi = x - z$ to the integral on the right-hand side above, we get

$$\begin{aligned} |\,f(x,t)\,| \;&\leqslant\; |||\,g\,|||_{\beta;\,[0,T]}\int_0^t\int_{\mathbb{R}^2}\frac{1}{4\pi(t-s)}\,e^{-|\xi|^2/4(t-s)}\,e^{-(|\xi-x|+|\xi|)/\beta}\,e^{|\xi|/\beta}\,\mathrm{d}\xi\,\mathrm{d}s\\ &\leqslant\; |||\,g\,|||_{\beta;\,[0,T]}\,e^{-|x|/\beta}\int_0^t\int_{\mathbb{R}^2}\frac{1}{4\pi(t-s)}\,e^{-|\xi|^2/4(t-s)}\,e^{|\xi|/\beta}\,\mathrm{d}\xi\,\mathrm{d}s. \end{aligned}$$

Now we let $2\eta = \xi/(t-s)^{1/2}$ and reduce the last estimate to

$$\begin{aligned} |\,f(x,t)\,| \;&\lesssim\; |||\,g\,|||_{\beta;\,[0,T]}\,e^{-|x|/\beta}\int_0^t\int_{\mathbb{R}^2} e^{-|\eta|^2+2(t-s)^{1/2}|\eta|/\beta}\,\mathrm{d}\eta\,\mathrm{d}s\\ &\lesssim\; |||\,g\,|||_{\beta;\,[0,T]}\,t\,e^{-|x|/\beta+2t/\beta^2}. \end{aligned} \tag{2.5}$$

The first derivatives of f can be represented as follows:

$$\nabla f(x,t) = 2^{-1}\int_0^t\int_{\mathbb{R}^2}\mathrm{G}(x-z,t-s)\,g(z,s)\,\frac{z-x}{t-s}\,\mathrm{d}z\,\mathrm{d}s. \tag{2.6}$$

Similarly to the above arguments for f, the following estimate holds for ∇f

$$
\begin{aligned}
|\nabla f(x,t)| &\lesssim \|\!|g|\!\|_{\beta;[0,T]} \int_0^t \int_{\mathbb{R}^2} e^{-|x-z|^2/4(t-s)}\, e^{-|z|/\beta} \frac{|x-z|}{(t-s)^2}\, \mathrm{d}z\, \mathrm{d}s \\
&\lesssim \|\!|g|\!\|_{\beta;[0,T]} \int_0^t \int_{\mathbb{R}^2} e^{-|\xi|^2/4(t-s)}\, e^{-|\xi-x|/\beta-|\xi|/\beta}\, e^{|\xi|/\beta} \frac{|\xi|}{(t-s)^2}\, \mathrm{d}\xi\, \mathrm{d}s \\
&\lesssim \|\!|g|\!\|_{\beta;[0,T]}\, e^{-|x|/\beta} \int_0^t \int_{\mathbb{R}^2} e^{-|\eta|^2+2(t-s)^{1/2}|\eta|/\beta} \frac{|\eta|}{(t-s)^{1/2}}\, \mathrm{d}\eta\, \mathrm{d}s \\
&\lesssim \|\!|g|\!\|_{\beta;[0,T]}\, t^{1/2}\, e^{-|x|/\beta+2t/\beta^2}.
\end{aligned} \tag{2.7}
$$

In light of (2.6), the second order derivatives of f can be represented as follows:

$$
\partial_{ij} f(x,t) = 2^{-1} \int_0^t \int_{\mathbb{R}^2} \mathrm{G}(x-z,t-s)\big[g(z,s)-g(x,s)\big]\left[\frac{(z_i-x_i)(z_j-x_j)}{2(t-s)^2} - \frac{\delta_{ij}}{t-s}\right] \mathrm{d}z\, \mathrm{d}s, \tag{2.8}
$$

where δ_{ij} is the Kronecker delta. Therefore we can estimate $\partial_{ij} f$ as shown below:

$$
\begin{aligned}
\big|\partial_{ij} f(x,t)\big| &\lesssim \int_0^t \int_{\mathbb{R}^2} \mathrm{G}(x-z,t-s)\,\big|g(z,s)-g(x,s)\big|^\theta \big|g(z,s)-g(x,s)\big|^{1-\theta} \left[\frac{|z-x|^2}{(t-s)^2} + \frac{1}{t-s}\right] \mathrm{d}z\, \mathrm{d}s \\
&\lesssim \max_{t\in[0,T]} \big[g(\cdot,t)\big]_\alpha^{1-\theta} \int_0^t \int_{\mathbb{R}^2} \mathrm{G}(z,t-s) \frac{|z|^2+(t-s)}{(t-s)^2} |z|^{(1-\theta)\alpha} \Big(\big|g(x-z,s)\big|^\theta + \big|g(x,s)\big|^\theta\Big) \\
&\lesssim_{\theta,\alpha} \max_{t\in[0,T]} \big[g(\cdot,t)\big]_\alpha^{1-\theta}\, \|\!|g|\!\|^\theta_{\beta;[0,T]}\, t^{(1-\theta)\alpha/2}\, e^{-\theta|x|/\beta+2t\theta^2/\beta^2}.
\end{aligned}
$$

The third inequality above holds by similar arguments as in the derivations of (2.5) and (2.7). The proof is then finished in light of (2.5), (2.7) and the last estimate. □

In the next lemma, we present an $\mathrm{L}^\infty \mathrm{C}^\beta$-estimate for the second order derivatives of f, where $\beta \in (0,\alpha)$ and $f = \Phi[g]$ as in Lemma 2.1. Since the proof is similar to the proof of Lemma 4.4 in [13], we omit it here.

Lemma 2.2. *Under the same assumptions for g as in Lemma 2.1, $\nabla^2 f$ lies in the space $\mathrm{L}^\infty\big([0,T];\mathrm{C}^\beta(\mathbb{R}^2)\big)$ for all $\beta \in (0,\alpha)$. Here $f = \Phi[g]$ is a solution of (2.2). Moreover $\nabla^2 f$ satisfies the estimate given below:*

$$
\max_{t\in[0,T]} \big[\nabla^2 f(\cdot,t)\big]_\beta \lesssim_{\alpha,\beta} \max_{t\in[0,T]} [g(\cdot,t)]_\alpha\, T^{\alpha/2-\beta/2}.
$$

As for the initial value problem for the homogeneous linear heat equation, we have

Lemma 2.3. *Suppose that $g \in \mathrm{C}^k(\mathbb{R}^2)$ with $k \in \mathbb{N}\cup\{0\}$. Moreover we assume that $\nabla^k g \in \mathrm{C}^*_\beta(\mathbb{R}^2)$ for some $\beta > 0$. With the function g, we define*

$$
\Psi[g](x,t) = \int_{\mathbb{R}^2} \mathrm{G}(x-z,t)\, g(z)\, \mathrm{d}z, \qquad \forall\, (x,t) \in \mathbb{R}^2 \times [0,T].
$$

Then $\Psi[g]$ is a solution of the following initial value problem:

$$
\begin{cases}
\partial_t F - \Delta F = 0, & \text{in } \mathbb{R}^2 \times [0,T]; \\
F = g\,, & \text{at } t = 0.
\end{cases}
$$

Moreover for all $T > 0$, $\Psi[g]$ satisfies the estimate given below:

$$
\|\!|\nabla^k \Psi[g]|\!\|_{\beta;[0,T]} \lesssim \|\!|\nabla^k g|\!\|_\beta\, e^{2T/\beta^2}.
$$

Proof. For simplicity we use F to denote the function $\Psi[g]$. By making derivatives k times, it holds

$$\nabla^k F(x,t) = \int_{\mathbb{R}^2} \mathrm{G}(x-z,t)\, \nabla^k g(z)\, \mathrm{d}z, \qquad \forall\, (x,t) \in \mathbb{R}^2 \times [0,T].$$

Since $\nabla^k g \in \mathrm{C}^*_\beta(\mathbb{R}^2)$, we then have

$$\big|\nabla^k F(x,t)\big| \;\leqslant\; \big|\!\big|\!\big|\nabla^k g\big|\!\big|\!\big|_\beta \int_{\mathbb{R}^2} \mathrm{G}(x-z,t)\, e^{-|z|/\beta}\, \mathrm{d}z.$$

The proof then follows by a similar argument as the derivation of (2.5). □

Now we consider some embedding properties associated with $\mathrm{C}^*_\beta[I]$ and $\mathrm{C}^*_\beta(\mathbb{R}^2)$.

Lemma 2.4. *For any $p \in [1,\infty)$ and $\beta > 0$, the space $\mathrm{C}^*_\beta(\mathbb{R}^2)$ is embedded into $\mathrm{L}^p(\mathbb{R}^2)$. Moreover for any $f \in \mathrm{C}^*_\beta(\mathbb{R}^2)$, we have*

$$\|f\|_p \;\lesssim_{p,\beta}\; |\!|\!| f |\!|\!|_\beta.$$

In the same fashion $\mathrm{C}^*_\beta[I]$ is embedded into the space $\mathrm{C}\big(I;\mathrm{L}^p(\mathbb{R}^2)\big)$. Here I is a finite time interval. For any $f \in \mathrm{C}^*_\beta[I]$, the following estimate is satisfied:

$$\|f\|_{\mathrm{L}^\infty(I;\mathrm{L}^p(\mathbb{R}^2))} \;\lesssim_{p,\beta}\; |\!|\!| f |\!|\!|_{\beta;I}.$$

Proof. The proof of the estimates in this lemma is simple in that f is exponentially decay at spatial infinity if $f \in \mathrm{C}^*_\beta[I]$ or $\mathrm{C}^*_\beta(\mathbb{R}^2)$. We only need show that $f(t,\cdot)$ is a continuous mapping from I to $\mathrm{L}^p(\mathbb{R}^2)$ if $f \in \mathrm{C}^*_\beta[I]$. Let t_n be an arbitrary sequence in I which converges to some $t_0 \in I$. Since $f \in \mathrm{C}^*_\beta[I]$, it holds

$$\big|f(t_n,x) - f(t_0,x)\big| \;\lesssim\; \big|f(t_n,x)\big|\, e^{|x|/\beta}\, e^{-|x|/\beta} + \big|f(t_0,x)\big|\, e^{|x|/\beta}\, e^{-|x|/\beta} \;\lesssim\; |\!|\!| f |\!|\!|_{\beta;I}\, e^{-|x|/\beta}.$$

The most-right-hand side above is L^p-integrable on $\mathbb{R}^2$. Thus the continuity of the function f and Lebesgue's dominated convergence theorem imply that $f(t_n,\cdot) \longrightarrow f(t_0,\cdot)$ in L^p, as $n \to \infty$. The proof is finished. □

This lemma combined with the Calderon-Zygmund estimate leads to the following result:

Lemma 2.5. *If $\omega \in \mathrm{C}^*_\beta(\mathbb{R}^2)$, then for all $p \in (2,\infty)$, $v = \mathrm{K} * \omega$ lies in $W^{1,p}(\mathbb{R}^2)$. Moreover by Morrey's inequality, we have*

$$\|v\|_{\mathrm{C}^{(p-2)/p}} \;\lesssim_p\; \|v\|_{1,p} \;\lesssim_p\; \|\omega\|_{2p/(p+2)} + \|\omega\|_p \;\lesssim_{p,\beta}\; |\!|\!|\omega|\!|\!|_\beta.$$

Here the first inequality is Morrey's inequality. The second inequality above is the Calderon-Zygmund estimate. The last inequality uses our Lemma 2.4.

In the end we study the continuity of $v = \mathrm{K} * \omega$ with $\omega \in \mathrm{C}^*_\beta[0,T]$.

Lemma 2.6. *If $\omega \in \mathrm{C}^*_\beta[0,T]$, then $v = \mathrm{K} * \omega$ is continuous on $\mathbb{R}^2 \times [0,T]$.*

Proof. Suppose that (x_0,t_0) is an arbitrary point on $\mathbb{R}^2 \times [0,T]$ and $\big\{(x_n,t_n)\big\} \subset \mathbb{R}^2 \times [0,T]$ is an arbitrary sequence which converges to (x_0,t_0). By the definition of v, we have

$$v(x_n,t_n) = \int_{\mathbb{R}^2} \mathrm{K}(z)\, \omega(x_n - z, t_n)\, \mathrm{d}z. \tag{2.9}$$

In light of $\omega \in \mathrm{C}^*_\beta[0,T]$, it holds

$$\big|\mathrm{K}(z)\,\omega(x_n-z,t_n)\big| \;\lesssim\; |z|^{-1}\,\big|\omega(x_n-z,t_n)\big|\, e^{|x_n-z|/\beta}\, e^{-|x_n-z|/\beta} \;\lesssim_{\beta,\,\max_n|x_n|}\; |\!|\!|\omega|\!|\!|_{\beta;[0,T]}\, |z|^{-1}\, e^{-|z|/\beta}.$$

Here we have used the boundedness of the sequence $\{x_n\}$. Since the function on the most-right-hand side above is integrable on $\mathbb{R}^2$ and ω is continuous on $\mathbb{R}^2 \times [0,T]$, then by (2.9) and the Lebesgue's dominated convergence theorem, we have $v(x_n,t_n) \to v(x_0,t_0)$ as $n \to \infty$. The proof is finished. □

III. EXISTENCE OF SHORT-TIME CLASSIC SOLUTIONS

In this section we prove Theorem 1.2.

III.1. SKETCH OF THE PROOF AND SOME PRELIMINARY LEMMAS

Our proof is based on a fixed point argument in the functional space X given below:

$$X := \Big\{ (\phi, \omega) \ : \ |||\, \phi - \phi_* \,|||_{2;\,1;\,[0,T]} \ + \ |||\, \nabla^3 \phi - \nabla^3 \phi_* \,|||_{2;\,[0,T]} \ + \ |||\, \omega - \omega_* \,|||_{1;\,2;\,[0,T]} \tag{3.1}$$

$$+ \ |||\, \nabla^2 \omega - \nabla^2 \omega_* \,|||_{4;\,[0,T]} \ \leqslant \ 1 \ \text{ and } \ (\phi, \omega)\,|_{t=0} = (\phi_0, \omega_0) \Big\},$$

where with the operator Ψ defined in Lemma 2.3, $(\phi_*, \omega_*) := \big(\Psi[\phi_0], \Psi[\omega_0] \big)$ is a solution of the following initial value problem:

$$\begin{cases} \partial_t \phi_* - \Delta \phi_* = 0, \ \ \partial_t \omega_* - \Delta \omega_* = 0, & \text{in } \mathbb{R}^2 \times (0, \infty); \\ \phi_*(\cdot, 0) = \phi_0(\cdot), \ \ \omega_*(\cdot, 0) = \omega_0(\cdot). \end{cases} \tag{3.2}$$

Since we are studying local existence of (1.2), T can be supposed to be as small as possible.

Now we sketch the proof and make some preliminary lemmas for later use. Letting (ϕ, ω) be an arbitrary element in X and $v = \mathrm{K} * \omega$, we denote by $(\psi, w) = \mathbf{S}(\phi, \omega)$ the solution of the following Cauchy problem:

$$\begin{cases} \partial_t \psi - \Delta \psi = \mathrm{F}(\phi, \omega) := \eta(|\phi|) \big[\,|\nabla \hat{\phi}|^2 \hat{\phi} - v \cdot \nabla \hat{\phi} \,\big], & \text{in } \mathbb{R}^2 \times (0, T); \\ \partial_t w - \Delta w = -v \cdot \nabla \omega - \nabla \times \nabla \cdot (\nabla \psi \odot \nabla \psi), & \text{in } \mathbb{R}^2 \times (0, T); \\ v = \mathrm{K} * \omega, \ \ \psi(0, \cdot) = \phi_0(\cdot), \ \ w(0, \cdot) = \omega_0(\cdot). \end{cases} \tag{3.3}$$

In (3.3) η is a non-negative smooth cut-off function defined on $\mathbb{R}_+$ which satisfies $\eta \equiv 1$ on $(1/2, \infty)$ and $\eta \equiv 0$ on $(0, 1/4)$. Moreover $\hat{\phi} = \phi / |\phi|$ is the normalized vector of ϕ. If the operator $\mathbf{S}$ has a fixed point in X, then by (3.3), the fixed point must solve the following initial value problem:

$$\begin{cases} \partial_t \phi - \Delta \phi = \mathrm{F}(\phi, \omega), & \text{in } \mathbb{R}^2 \times (0, T); \\ \partial_t \omega - \Delta \omega = -v \cdot \nabla \omega - \nabla \times \nabla \cdot (\nabla \phi \odot \nabla \phi), & \text{in } \mathbb{R}^2 \times (0, T); \\ v = \mathrm{K} * \omega, \ \ \phi(0, \cdot) = \phi_0(\cdot), \ \ \omega(0, \cdot) = \omega_0(\cdot). \end{cases} \tag{3.4}$$

A simple maximum principle yields that solutions of (3.4) with the images of ϕ_0 in $\mathbb{S}^2$ is a solution of (1.2). Therefore the proof of Theorem 1.2 is then reduced to show that $\mathbf{S}$ is a contraction mapping from X to itself. To do so, we substract (3.2) from (3.3) and get the following Cauchy problem satisfied by $(\psi - \phi_*, w - \omega_*)$:

$$\begin{cases} \partial_t (\psi - \phi_*) - \Delta (\psi - \phi_*) = \mathrm{F}(\phi, \omega), & \text{in } \mathbb{R}^2 \times (0, T); \\ \partial_t (w - \omega_*) - \Delta (w - \omega_*) = -v \cdot \nabla \omega - \nabla \times \nabla \cdot \big(\nabla \psi \odot \nabla \psi \big), & \text{in } \mathbb{R}^2 \times (0, T); \\ \psi(0, \cdot) - \phi_*(0, \cdot) = 0, \ \ w(0, \cdot) - \omega_*(0, \cdot) = 0. \end{cases} \tag{3.5}$$

Now we should prove $(\psi, w) \in X$. Thus we need

Lemma 3.1. *There exists a positive constant M so that for all $(\phi,\omega)\in X$, we have*

$$\left|\!\left|\!\left| \mathrm{F}(\phi,\omega) \right|\!\right|\!\right|_{1;1} + \left|\nabla^2 \mathrm{F}(\phi,\omega)\right|_0 \;\leqslant\; M.$$

Here M depends on $|||\cdot|||_{4;1}$-norm of $\phi_0 - e$ and $|||\cdot|||_{2;2}$-norm of ω_0.

In this lemma and the remainings of this section, if the space-time is $\mathbb{R}^2\times[0,T]$, we always use $|\cdot|_0$ to simply denote the L^∞-norm of a given quantity on $\mathbb{R}^2\times[0,T]$. To show that $\mathbf{S}$ is a contraction mapping, the following lemma is required:

Lemma 3.2. *There exists a positive constant M depending only on the $|||\cdot|||_{4;1}$-norm of ϕ_0-e and $|||\cdot|||_{2;2}$-norm of ω_0 such that for all $(\phi_j,\omega_j)\in X$ $(j=1,2)$, we have*

$$\begin{cases} \left|\mathrm{F}(\phi_1,\omega_1)-\mathrm{F}(\phi_2,\omega_2)\right| \;\lesssim_M\; \sum_{i=0}^{1}\left|\nabla^i\phi_1-\nabla^i\phi_2\right| + \left(\sum_{i=1}^{2}\left|\nabla\phi_i\right|\right)\left|v_1-v_2\right|; \\[2ex] \left|\nabla\mathrm{F}(\phi_1,\omega_1)-\nabla\mathrm{F}(\phi_2,\omega_2)\right| \;\lesssim_M\; \sum_{i=0}^{2}\left|\nabla^i\phi_1-\nabla^i\phi_2\right| + \left(\sum_{i,j=1}^{2}\left|\nabla^i\phi_j\right|\right)\left(\sum_{i=0}^{1}\left|\nabla^i v_1-\nabla^i v_2\right|\right) \\[2ex] \left|\nabla^2\mathrm{F}(\phi_1,\omega_1)-\nabla^2\mathrm{F}(\phi_2,\omega_2)\right|_0 \;\lesssim_M\; \sum_{i=0}^{3}\left|\nabla^i\phi_1-\nabla^i\phi_2\right|_0 + \sum_{i=0}^{2}\left|\nabla^i v_1-\nabla^i v_2\right|_0. \end{cases}$$

Here $v_j = \mathrm{K}*\omega_j$ $(j=1,2)$ are two velocity fields recovered by the Biot-Savart law.

In the remaining of this section we finish the proof of Lemma 3.1. With (3.8) and (3.10) below, the proof of Lemma 3.2 can be easily obtained and hence is omitted for brevity.

Proof of Lemma 3.1. Under the assumptions made for (ϕ_0,ω_0) in Theorem 1.2, Lemma 2.3 implies that

$$|||\phi_* - e|||_{4;1;[0,T]} \;\lesssim\; |||\phi_0-e|||_{4;1}\, e^{2T} \qquad \text{and} \qquad |||\omega_*|||_{2;2;[0,T]} \;\lesssim\; |||\omega_0|||_{2;2}\, e^{T/2}. \tag{3.6}$$

In view of the definition of the space X in (3.1), we have $\omega\in \mathrm{C}_4^{*,2}[0,T]$ and $\phi\in\mathrm{C}_2^{*,3}[0,T]$, which yield, by Lemma 2.6, the continuity of v, ∇v and $\nabla^2 v$ on $\mathbb{R}^2\times[0,T]$. Therefore we know that $\nabla^i\mathrm{F}(\phi,\omega)$ $(i=0,1,2)$ are all continuous on $\mathbb{R}^2\times[0,T]$. We are left to show the estimate in Lemma 3.1.

For any $p>2$ and $t\in[0,T]$, it holds

$$\|\nabla^i v(\cdot,t)\|_\infty \;\lesssim_p\; \|\nabla^i\omega(\cdot,t)\|_1 + \|\nabla^i\omega(\cdot,t)\|_p, \qquad i=0,1,2.$$

Taking supremum over all $t\in[0,T]$ and using Lemma 2.4, we can reduce the above estimates to

$$\left|\nabla^i v\right|_0 \;\lesssim\; \left|\!\left|\!\left|\nabla^i\omega\right|\!\right|\!\right|_{4;[0,T]}, \qquad i=0,1,2. \tag{3.7}$$

Employing (3.6) and the definition of X in (3.1), we can show

$$|||\phi-e|||_{2;1;[0,T]} + \left|\!\left|\!\left|\nabla^3\phi\right|\!\right|\!\right|_{2;[0,T]} \;\lesssim\; |||\phi_0-e|||_{4;1} + 1 \tag{3.8}$$

and

$$|||\omega|||_{1;2;[0,T]} + \left|\!\left|\!\left|\nabla^2\omega\right|\!\right|\!\right|_{4;[0,T]} \;\lesssim\; |||\omega_0|||_{2;2} + 1. \tag{3.9}$$

Thus by (3.7) and (3.9), it holds

$$\sum_{i=0}^{2}\left|\nabla^i v\right|_0 \;\leqslant\; M. \tag{3.10}$$

Here and in what follows M is a constant depending only on the $|||\cdot|||_{4;1}$-norm of $\phi_0 - e$ and $|||\cdot|||_{2;2}$-norm of ω_0. In light of the definition of $\mathrm{F}(\phi,\omega)$ in (3.3), by (3.8), (3.10) and direct calculations, we can show that

$$\begin{cases} \left|\mathrm{F}(\phi,\omega)\right| \lesssim \left|\nabla\phi\right|^2 + \left|v\right|\left|\nabla\phi\right|, \\ \left|\nabla\mathrm{F}(\phi,\omega)\right| \lesssim \left[|v| + |\nabla\phi|\right]\left[|\nabla\phi|^2 + |\nabla^2\phi|\right] + \left[|v|^2 + |\nabla v|\right]|\nabla\phi|, \\ \left|\nabla^2\mathrm{F}(\phi,\omega)\right|_0 \leqslant M, \end{cases} \tag{3.11}$$

Using (3.8), (3.10) and the first estimate in (3.11), we get

$$\left|\mathrm{F}(\phi,\omega)\right| e^{|x|} \lesssim \left|\nabla\phi\right|^2 e^{|x|} + \left|v\right|_0 \left|\nabla\phi\right| e^{|x|} \leqslant M, \qquad \forall\,(x,t) \in \mathbb{R}^2 \times [0,T].$$

Taking supreme over $\mathbb{R}^2 \times [0,T]$, we obtain the desired uniform boundedness of $\mathrm{F}(\phi,\omega)$. Same arguments can be applied to show that $\nabla\mathrm{F}(\phi,\omega)$ is uniformly bounded from above by M in $\mathrm{C}_1^*[0,T]$. Here one just needs (3.8), (3.10) and the second estimate in (3.11). □

III.2. PROOF OF THEOREM 1.2

Now we proceed to the proof of Theorem 1.2.

Proof of Theorem 1.2: In the proof we still use M to denote a large positive constant depending only on the $|||\cdot|||_{4;1}$-norm of $\phi_0 - e$ and $|||\cdot|||_{2;2}$-norm of ω_0.

Step 1. Let (ϕ,ω) be an arbitrary element in X. $(\psi,w) = \mathbf{S}(\phi,\omega)$ is the solution of (3.5). The $|||\cdot|||_{1;1}$-norm of $\psi - \phi_*$ can be estimated by Lemma 2.1. With (2.3) and the first equation in (3.5), one can show that

$$|||\,\psi - \phi_*\,|||_{1;[0,T]} + T^{1/2}\,|||\,\nabla\psi - \nabla\phi_*\,|||_{1;[0,T]} \lesssim T\,|||\,\mathrm{F}(\phi,\omega)\,|||_{1;[0,T]}.$$

Applying Lemma 3.1 to the right-hand side above implies

$$|||\,\psi - \phi_*\,|||_{1;[0,T]} + T^{1/2}\,|||\,\nabla\psi - \nabla\phi_*\,|||_{1;[0,T]} \lesssim_M T. \tag{3.12}$$

Making spatial derivative one more time on both sides of the first equation in (3.5), by Lemma 2.1 and Lemma 3.1, we can derive that

$$|||\,\nabla\psi - \nabla\phi_*\,|||_{1;[0,T]} + T^{1/2}\,|||\,\nabla^2\psi - \nabla^2\phi_*\,|||_{1;[0,T]} \lesssim T\,|||\,\nabla\mathrm{F}(\phi,\omega)\,|||_{1;[0,T]} \lesssim_M T. \tag{3.13}$$

Moreover in light of (2.4), $\nabla^3\psi - \nabla^3\phi_*$ can be estimated as follows:

$$\begin{aligned} |||\,\nabla^3\psi - \nabla^3\phi_*\,|||_{2;[0,T]} &\lesssim_\alpha T^{\alpha/4} \max_{t\in[0,T]} \left[\nabla\mathrm{F}(\phi,\omega)(\cdot,t)\right]_\alpha^{1/2} |||\,\nabla\mathrm{F}(\phi,\omega)\,|||_{1;[0,T]}^{1/2} \\ &\lesssim_{\alpha,M} T^{\alpha/4} \max_{t\in[0,T]} \left[\nabla\mathrm{F}(\phi,\omega)(\cdot,t)\right]_\alpha^{1/2}. \end{aligned}$$

Here we take $\theta = 1/2$ and $\beta = 1$ in (2.4). α is a constant in $(0,1)$. In light of Lemma 3.1, by interpolation inequality, we can show that

$$\max_{t\in[0,T]} \left[\nabla\mathrm{F}(\phi,\omega)(\cdot,t)\right]_\alpha \lesssim \left|\nabla\mathrm{F}(\phi,\omega)\right|_0 + \left|\nabla^2\mathrm{F}(\phi,\omega)\right|_0 \leqslant M. \tag{3.14}$$

Thus the above two estimates imply that

$$|||\,\nabla^3\psi - \nabla^3\phi_*\,|||_{2;[0,T]} \lesssim_{\alpha,M} T^{\alpha/4}. \tag{3.15}$$

Combining this estimate with the first estimate in (3.6), we have

$$\left\| \nabla^3 \psi \right\|_{2;[0,T]} \leqslant M. \tag{3.16}$$

Furthermore by (3.13) and the first estimate in (3.6), the following boundedness holds

$$\left\| \nabla \psi \right\|_{1;[0,T]} + \left\| \nabla^2 \psi \right\|_{1;[0,T]} \leqslant M. \tag{3.17}$$

In light of (3.9)-(3.10) and (3.16)-(3.17), one can easily show that

$$\left\| v \cdot \nabla \omega + \nabla \times \nabla \cdot \left(\nabla \psi \odot \nabla \psi \right) \right\|_{2;[0,T]} \leqslant M. \tag{3.18}$$

Applying this estimate and Lemma 2.1 to the second equation in (3.5) yields

$$\left\| w - \omega_* \right\|_{2;[0,T]} + T^{1/2} \left\| \nabla w - \nabla \omega_* \right\|_{2;[0,T]} \lesssim_M T. \tag{3.19}$$

Taking one more spatial derivative on both sideds of the first equation in (3.5), by (3.14) and Lemma 2.2, we can show for any $\beta \in (0, \alpha)$ that

$$\max_{t \in [0,T]} \left[\nabla^3 \psi(\cdot, t) - \nabla^3 \phi_*(\cdot, t) \right]_\beta \lesssim_{\alpha,\beta} \max_{t \in [0,T]} \left[\nabla \mathrm{F}(\phi, \omega)(\cdot, t) \right]_\alpha T^{\alpha/2 - \beta/2} \lesssim_{\alpha,\beta,M} T^{\alpha/2-\beta/2}.$$

By an interpolation inequality, the first estimate in (3.6) yields

$$\max_{t \in [0,T]} \left[\nabla^3 \phi_*(\cdot, t) \right]_\beta \leqslant M.$$

Thus the above two estimates imply that

$$\max_{t \in [0,T]} \left[\nabla^3 \psi(\cdot, t) \right]_\beta \lesssim_\beta M. \tag{3.20}$$

In light of this estimate, (3.9)-(3.10) and (3.16)-(3.17), by interpolation inequalities, it can be shown that

$$\max_{t \in [0,T]} \left[v \cdot \nabla \omega + \nabla \times \nabla \cdot \left(\nabla \psi \odot \nabla \psi \right) \right]_\beta \lesssim_\beta M. \tag{3.21}$$

Using this estimate, (3.18) and (2.4) in Lemma 2.1, $\nabla^2 w - \nabla^2 \omega_*$ can be estimated as follows:

$$\left\| \nabla^2 w - \nabla^2 \omega_* \right\|_{4;[0,T]} \tag{3.22}$$

$$\lesssim_\beta \max_{t \in [0,T]} \left[v \cdot \nabla \omega + \nabla \times \nabla \cdot \left(\nabla \psi \odot \nabla \psi \right) \right]_\beta^{1/2} \left\| v \cdot \nabla \omega + \nabla \times \nabla \cdot \left(\nabla \psi \odot \nabla \psi \right) \right\|_{2;[0,T]}^{1/2} T^{\beta/4} \lesssim_{\beta,M} T^{\beta/4}.$$

Here we used the second equation in (3.5). In light of (3.12)-(3.13), (3.15), (3.19) and (3.22), if we take T depending on M and β to be small enough, then $(\psi, w) \in X$. This shows that $\mathbf{S}$ is an operator from X to itself.

Step 2. This step is devoted to showing that $\mathbf{S}$ is a contraction mapping. In the remaining of this step we let (ϕ_1, ω_1) and (ϕ_2, ω_2) be two arbitrary elements in X. For $j = 1, 2$, we denote by v_j the vector field $\mathrm{K} * \omega_j$. If $(\psi_j, w_j) = \mathbf{S}(\phi_j, \omega_j)$ $(j = 1, 2)$, then by (3.5) it holds

$$\begin{cases} \partial_t (\psi_1 - \psi_2) - \Delta (\psi_1 - \psi_2) = \mathrm{F}(\phi_1, \omega_1) - \mathrm{F}(\phi_2, \omega_2), & \text{in } \mathbb{R}^2 \times (0, T); \\ \partial_t (w_1 - w_2) - \Delta (w_1 - w_2) = -\left[v_1 \cdot \nabla \omega_1 - v_2 \cdot \nabla \omega_2 \right] & \text{in } \mathbb{R}^2 \times (0, T); \\ \qquad -\left[\nabla \times \nabla \cdot \left(\nabla \psi_1 \odot \nabla \psi_1 \right) - \nabla \times \nabla \cdot \left(\nabla \psi_2 \odot \nabla \psi_2 \right) \right]; & \\ \psi_1(0, \cdot) - \psi_2(0, \cdot) = 0, \quad w_1(0, \cdot) - w_2(0, \cdot) = 0. & \end{cases} \tag{3.23}$$

The $|||\cdot|||_{1;1}$-norm of $\psi_1 - \psi_2$ can be estimated by Lemma 2.1. With (2.3) and the first equation in (3.23), one can show that

$$|||\,\psi_1 - \psi_2\,|||_{1;[0,T]} + T^{1/2}\,|||\,\nabla\psi_1 - \nabla\psi_2\,|||_{1;[0,T]} \lesssim T\,|||\,\mathrm{F}(\phi_1,\omega_1) - \mathrm{F}(\phi_2,\omega_2)\,|||_{1;[0,T]}.$$

Using the first estimate in Lemma 3.2 and (3.8), we have

$$|||\,\mathrm{F}(\phi_1,\omega_1) - \mathrm{F}(\phi_2,\omega_2)\,|||_{1;[0,T]} \quad \lesssim_M \quad |||\,\phi_1 - \phi_2\,|||_{1;1;[0,T]} + \big|v_1 - v_2\big|_0.$$

The last two estimates imply that

$$|||\,\psi_1 - \psi_2\,|||_{1;[0,T]} + T^{1/2}\,|||\,\nabla\psi_1 - \nabla\psi_2\,|||_{1;[0,T]} \lesssim_M T\left[\,|||\,\phi_1 - \phi_2\,|||_{1;1;[0,T]} + \big|v_1 - v_2\big|_0\right].$$

Moreover by Lemma 2.5, this estimate can be reduced to

$$\begin{aligned} |||\,\psi_1 - \psi_2\,|||_{1;[0,T]} + T^{1/2}\,|||\,\nabla\psi_1 - \nabla\psi_2\,|||_{1;[0,T]} \quad &\lesssim_M \quad T\left[\,|||\,\phi_1 - \phi_2\,|||_{1;1;[0,T]} + |||\,\omega_1 - \omega_2\,|||_{2;[0,T]}\right] \\ &\leqslant \quad T\,|||\,(\phi_1 - \phi_2, \omega_1 - \omega_2)\,|||_X. \end{aligned} \tag{3.24}$$

Here we used $|||\,(\phi_1 - \phi_2, \omega_1 - \omega_2)\,|||_X$ to simply denote the sum

$$|||\,\phi_1 - \phi_2\,|||_{2;1;[0,T]} + \left|\left|\left|\,\nabla^3\phi_1 - \nabla^3\phi_2\,\right|\right|\right|_{2;[0,T]} + |||\,\omega_1 - \omega_2\,|||_{1;2;[0,T]} + \left|\left|\left|\,\nabla^2\omega_1 - \nabla^2\omega_2\,\right|\right|\right|_{4;[0,T]}.$$

Making spatial derivative one more time on both sides of the first equation in (3.23), by Lemma 2.1, we have

$$|||\,\nabla\psi_1 - \nabla\psi_2\,|||_{1;[0,T]} + T^{1/2}\left|\left|\left|\,\nabla^2\psi_1 - \nabla^2\psi_2\,\right|\right|\right|_{1;[0,T]} \quad \lesssim \quad T\,|||\,\nabla\mathrm{F}(\phi_1,\omega_1) - \nabla\mathrm{F}(\phi_2,\omega_2)\,|||_{1;[0,T]}.$$

In light of the second estimate in Lemma 3.2, by Lemma 2.5, it holds

$$\begin{aligned} |||\,\nabla\mathrm{F}(\phi_1,\omega_1) - \nabla\mathrm{F}(\phi_2,\omega_2)\,|||_{1;[0,T]} \quad &\lesssim_M \quad |||\,\phi_1 - \phi_2\,|||_{2;1;[0,T]} + \sum_{i=0}^{1}\big|\nabla^i v_1 - \nabla^i v_2\big|_0 \\ &\lesssim_M \quad |||\,\phi_1 - \phi_2\,|||_{2;1;[0,T]} + |||\,\omega_1 - \omega_2\,|||_{1;2;[0,T]}. \end{aligned} \tag{3.25}$$

The last two estimates then yield

$$|||\,\nabla\psi_1 - \nabla\psi_2\,|||_{1;[0,T]} + T^{1/2}\left|\left|\left|\,\nabla^2\psi_1 - \nabla^2\psi_2\,\right|\right|\right|_{1;[0,T]} \quad \lesssim_M \quad T\,|||\,(\phi_1 - \phi_2, \omega_1 - \omega_2)\,|||_X. \tag{3.26}$$

The third order derivatives of $\psi_1 - \psi_2$ can be estimated by (2.4) and (3.25) as follows:

$$\begin{aligned} \left|\left|\left|\,\nabla^3\psi_1 - \nabla^3\psi_2\,\right|\right|\right|_{2;[0,T]} \qquad &\lesssim_{\alpha,M} \\ & T^{\alpha/4}\max_{t\in[0,T]}\big[\nabla\mathrm{F}(\phi_1,\omega_1)(\cdot,t) - \nabla\mathrm{F}(\phi_2,\omega_2)(\cdot,t)\big]_\alpha^{1/2}\,|||\,(\phi_1 - \phi_2, \omega_1 - \omega_2)\,|||_X^{1/2}. \end{aligned} \tag{3.27}$$

Here we take $\theta = 1/2$ and $\beta = 1$ in (2.4). α is a constant in $(0,1)$. In light of the second and third estimates in Lemma 3.2, by interpolation inequality, we can show that

$$\begin{aligned} \max_{t\in[0,T]}\big[\nabla\mathrm{F}(\phi_1,\omega_1)(\cdot,t) - \nabla\mathrm{F}(\phi_2,\omega_2)\big]_\alpha \quad &\lesssim \quad \sum_{i=1}^{2}\big|\nabla^i\mathrm{F}(\phi_1,\omega_1) - \nabla^i\mathrm{F}(\phi_2,\omega_2)\big|_0 \\ &\lesssim_M \quad \sum_{i=0}^{3}\big|\nabla^i\phi_1 - \nabla^i\phi_2\big|_0 + \sum_{i=0}^{2}\big|\nabla^i v_1 - \nabla^i v_2\big|_0. \end{aligned}$$

By Lemma 2.5, this estimate can be reduced to

$$\max_{t\in[0,T]} \big[\nabla \mathrm{F}(\phi_1,\omega_1)(\cdot,t) - \nabla \mathrm{F}(\phi_2,\omega_2)\big]_\alpha \quad \lesssim_M \quad \|\!|(\phi_1-\phi_2,\omega_1-\omega_2)|\!\|_X. \tag{3.28}$$

Thus (3.27)-(3.28) imply that

$$\big\|\!\big| \nabla^3\psi_1 - \nabla^3\psi_2 \big|\!\big\|_{2;[0,T]} \quad \lesssim_{\alpha,M} \quad T^{\alpha/4}\, \|\!|(\phi_1-\phi_2,\omega_1-\omega_2)|\!\|_X. \tag{3.29}$$

Direct calculations show that

$$\begin{aligned}
&\big\|\!\big| \nabla\times\nabla\cdot\big(\nabla\psi_1\odot\nabla\psi_1\big) - \nabla\times\nabla\cdot\big(\nabla\psi_2\odot\nabla\psi_2\big) \big|\!\big\|_{2;[0,T]} \\
&\lesssim \; \big|\nabla\psi_1\big|_0 \,\big\|\!\big|\nabla^3\psi_1 - \nabla^3\psi_2\big|\!\big\|_{2;[0,T]} \\
&+ \; \big|\nabla^3\psi_2\big|_0 \,\|\!|\nabla\psi_1-\nabla\psi_2|\!\|_{2;[0,T]} + \big[\,\big|\nabla^2\psi_1\big|_0 + \big|\nabla^2\psi_2\big|_0\big]\, \big\|\!\big|\nabla^2\psi_1-\nabla^2\psi_2\big|\!\big\|_{2;[0,T]}.
\end{aligned}$$

Therefore by (3.16)-(3.17), (3.26) and (3.29), this estimate can be reduced to

$$\big\|\!\big| \nabla\times\nabla\cdot\big(\nabla\psi_1\odot\nabla\psi_1\big) - \nabla\times\nabla\cdot\big(\nabla\psi_2\odot\nabla\psi_2\big) \big|\!\big\|_{2;[0,T]} \quad \lesssim_{\alpha,M} \quad T^{\alpha/4}\, \|\!|(\phi_1-\phi_2,\omega_1-\omega_2)|\!\|_X.$$

On the other hand using Lemma 2.5 and (3.9)-(3.10) yields

$$\begin{aligned}
\|\!|v_1\cdot\nabla\omega_1 - v_2\cdot\nabla\omega_2|\!\|_{2;[0,T]} \quad &\lesssim \quad \big|v_1-v_2\big|_0\, \|\!|\nabla\omega_1|\!\|_{2;[0,T]} + \big|v_2\big|_0\, \|\!|\nabla\omega_1-\nabla\omega_2|\!\|_{2;[0,T]} \\
&\lesssim \quad \|\!|\omega_1-\omega_2|\!\|_{2;[0,T]}\, \|\!|\nabla\omega_1|\!\|_{2;[0,T]} + \big|v_2\big|_0\, \|\!|\nabla\omega_1-\nabla\omega_2|\!\|_{2;[0,T]} \\
&\lesssim_M \quad \|\!|\omega_1-\omega_2|\!\|_{1;2;[0,T]}.
\end{aligned}$$

The last two estimates then imply that

$$\|\!|\mathrm{R.H.S.}|\!\|_{2;[0,T]} \quad \lesssim_M \quad \|\!|(\phi_1-\phi_2,\omega_1-\omega_2)|\!\|_X. \tag{3.30}$$

Here we used R.H.S. to simply denote the right-hand side of the second equation in (3.23). Applying this estimate and Lemma 2.1 to the second equation in (3.23), we get

$$\|\!|w_1-w_2|\!\|_{2;[0,T]} + T^{1/2}\, \|\!|\nabla w_1-\nabla w_2|\!\|_{2;[0,T]} \quad \lesssim_M \quad T\, \|\!|(\phi_1-\phi_2,\omega_1-\omega_2)|\!\|_X. \tag{3.31}$$

Taking one more spatial derivative on both sideds of the first equation in (3.23), by (3.28) and Lemma 2.2, we can show for any $\beta\in(0,\alpha)$ that

$$\begin{aligned}
\max_{t\in[0,T]} \big[\nabla^3\psi_1(\cdot,t) - \nabla^3\psi_2(\cdot,t)\big]_\beta \quad &\lesssim_{\alpha,\beta} \quad \max_{t\in[0,T]} \big[\nabla \mathrm{F}(\phi_1,\omega_1)(\cdot,t) - \nabla \mathrm{F}(\phi_2,\omega_2)(\cdot,t)\big]_\alpha\, T^{\alpha/2-\beta/2} \\
&\lesssim_{\alpha,\beta,M} \quad T^{\alpha/2-\beta/2}\, \|\!|(\phi_1-\phi_2,\omega_1-\omega_2)|\!\|_X.
\end{aligned}$$

Using this estimate, (3.16)-(3.17), (3.20), (3.26) and (3.29), by interpolation inequalities, we have

$$\max_{t\in[0,T]} \big[\nabla\times\nabla\cdot\big(\nabla\psi_1\odot\nabla\psi_1\big) - \nabla\times\nabla\cdot\big(\nabla\psi_2\odot\nabla\psi_2\big)\big]_\beta \lesssim_{\alpha,\beta,M} T^{\gamma}\, \|\!|(\phi_1-\phi_2,\omega_1-\omega_2)|\!\|_X,$$

where γ is a positive constant depending on α and β. The Hölder estimate for $v_1\cdot\nabla\omega_1 - v_2\cdot\nabla\omega_2$ can be estimated as follows:

$$\begin{aligned}
\max_{t\in[0,T]} \big[v_1\cdot\nabla\omega_1 - v_2\cdot\nabla\omega_2\big]_\beta \quad &\lesssim \quad \big|\nabla\omega_1\big|_0 \max_{t\in[0,T]} \big[v_1-v_2\big]_\beta + \big|v_1-v_2\big|_0 \max_{t\in[0,T]} \big[\nabla\omega_1\big]_\beta \\
&+ \quad \big|\nabla\omega_1-\nabla\omega_2\big|_0 \max_{t\in[0,T]} \big[v_2\big]_\beta + \big|v_2\big|_0 \max_{t\in[0,T]} \big[\nabla\omega_1-\nabla\omega_2\big]_\beta \\
&\lesssim_M \quad \|\!|\omega_1-\omega_2|\!\|_{1;2;[0,T]} + \big\|\!\big|\nabla^2\omega_1-\nabla^2\omega_2\big|\!\big\|_{4;[0,T]}.
\end{aligned}$$

To derive the above estimate, we used (3.9)-(3.10), Lemma 2.5 and various interpolation inequalities. Combining the last two estimates, one can easily show that

$$\max_{t\in[0,T]} \big[\text{R.H.S.}\big]_\beta \quad \lesssim_{\beta,M} \quad \| (\phi_1 - \phi_2, \omega_1 - \omega_2) \|_X .$$

In light of this estimate and (3.30), the following estimate holds by (2.4) in Lemma 2.1:

$$\begin{aligned} \| \nabla^2 w_1 - \nabla^2 w_2 \|_{4;[0,T]} \quad &\lesssim_\beta \quad \max_{t\in[0,T]} \big[\text{R.H.S.}\big]_\beta^{1/2} \, \|\text{R.H.S.}\|_{2;[0,T]}^{1/2} \, T^{\beta/4} \\ &\lesssim_{\beta,M} \quad T^{\beta/4} \, \| (\phi_1 - \phi_2, \omega_1 - \omega_2) \|_X . \end{aligned} \tag{3.32}$$

By (3.24), (3.26), (3.29) and (3.31)-(3.32), it holds

$$\| (\psi_1 - \psi_2, w_1 - w_2) \|_X \quad \lesssim_{\beta,M} \quad T^{\beta/4} \, \| (\phi_1 - \phi_2, \omega_1 - \omega_2) \|_X .$$

Therefore $\mathbf{S}$ is a contraction mapping from X to itself, provided that T is small enough.

Step 3. Now we choose T to be small enough. By the contraction mapping theorem, $\mathbf{S}$ admits a fixed point in X. Denoting by (ϕ, ω) the fixed point, we know that (ϕ, ω) is a solution of (3.4). Since $\phi \in \mathrm{C}_1^{*,2}[0,T]$, by Lemma 3.1 and the first equation in (3.4), we have $\partial_t \phi \in \mathrm{C}_1^*[0,T]$. It then turns out that

$$1 - |\phi(x,t)| \quad \leqslant \quad |\phi(x,t) - \phi_0| \quad \leqslant \quad \int_0^t |\partial_s \phi(x,s)| \, \mathrm{d}s \quad \leqslant \quad t \, \| \partial_t \phi \|_{1;[0,T]}, \qquad \forall\, (x,t) \in \mathbb{R}^2 \times (0,T).$$

If $t < T_* < T$, where T_* is sufficiently small, then $|\phi(x,t)| > 1/2$, for all $(x,t) \in \mathbb{R}^2 \times \big(0, T_*\big)$. In light that $\eta \equiv 1$ on $(1/2, \infty)$, the first equation in (3.4) can then be reduced to

$$\partial_t \phi - \Delta \phi \;=\; \big|\nabla \hat{\phi}\big|^2 \hat{\phi} - v \cdot \nabla \hat{\phi}, \qquad \text{on } \mathbb{R}^2 \times \big(0, T_*\big).$$

On the domain $\mathbb{R}^2 \times (0, T_*)$, this equation yields

$$\partial_t \rho - \Delta \rho \;=\; -2 \big|\nabla \phi - \nabla \hat{\phi}\big|^2, \qquad \text{where } \rho = \big|\phi - \hat{\phi}\big|^2.$$

A standard maximal principle implies that $\rho \equiv 0$ on $\mathbb{R}^2 \times \big(0, T_*\big)$. In other words on $\mathbb{R}^2 \times \big(0, T_*\big)$, (ϕ, ω) is a solution of (1.2) with $|\phi| \equiv 1$.

Step 4. In this step we show that $(\phi, \omega) \in \mathrm{C}_1^{*,4}[0,T_*] \times \mathrm{C}_2^{*,2}[0,T_*]$. Taking spatial derivative one more time on the both sides of the first equation in (1.2) and using ϕ_* in (3.2), we have

$$\partial_t \big(\partial_j \phi - \partial_j \phi_*\big) - \Delta \big(\partial_j \phi - \partial_j \phi_*\big) = -\partial_j v \cdot \nabla \phi - v \cdot \nabla \partial_j \phi + 2 \big(\nabla \phi : \nabla \partial_j \phi\big) \phi + \big|\nabla \phi\big|^2 \partial_j \phi. \tag{3.33}$$

In light of (3.14) and Lemma 3.1, by Lemma 2.1, it holds $\nabla^3 \phi - \nabla^3 \phi_* \in \mathrm{C}_\beta^*[0,T_*]$ for all $\beta \in (1,\infty)$. This result and the first estimate in (3.6) imply that $\nabla^3 \phi \in \mathrm{C}_\beta^*[0,T_*]$ for all $\beta \in (1,\infty)$. Taking one more spatial derivative on both sides of (3.33) and using ϕ_* in (3.2), we have

$$\partial_t \big(\partial_i \partial_j \phi - \partial_i \partial_j \phi_*\big) - \Delta \big(\partial_i \partial_j \phi - \partial_i \partial_j \phi_*\big) = -\partial_i \partial_j v \cdot \nabla \phi - v \cdot \nabla \partial_i \partial_j \phi + 2 \big(\nabla \phi : \nabla \partial_i \partial_j \phi\big) \phi + \text{l.o.t.}, \tag{3.34}$$

where l.o.t. is a quantity containing all the lower order terms on the right-hand side of (3.34). It can also be shown that l.o.t. lies in the space $\mathrm{C}_1^*[0,T_*]$. Applying (2.3) to the above equation, we obtain

$$\begin{aligned} \| \nabla \partial_i \partial_j \phi - \nabla \partial_i \partial_j \phi_* \|_{\beta;[0,T_*]} \quad &\lesssim \quad T_*^{1/2} \Big(|v|_{0;[0,T_*]} + |\nabla \phi|_{0;[0,T_*]} \Big) \| \nabla \partial_i \partial_j \phi \|_{\beta;[0,T_*]} \\ &+ \quad |\nabla^2 v|_{0;[0,T_*]} \, \| \nabla \phi \|_{\beta;[0,T_*]} + \| \text{l.o.t.} \|_{\beta;[0,T_*]}, \quad \beta \in (1,\infty). \end{aligned}$$

Employing (3.8) and (3.10), we can reduce the last estimate to

$$\begin{aligned}\|\!|\nabla \partial_i \partial_j \phi \|\!|_{\beta;[0,T_*]} &\lesssim \|\!|\nabla \partial_i \partial_j \phi_* \|\!|_{\beta;[0,T_*]} + M T_*^{1/2} \|\!|\nabla \partial_i \partial_j \phi \|\!|_{\beta;[0,T_*]} + M \|\!|\nabla \phi \|\!|_{\beta;[0,T_*]} + \|\!|\mathrm{l.o.t.}\|\!|_{\beta;[0,T_*]} \\ &\lesssim \|\!|\nabla \partial_i \partial_j \phi_* \|\!|_{1;[0,T_*]} + M T_*^{1/2} \|\!|\nabla \partial_i \partial_j \phi \|\!|_{\beta;[0,T_*]} + M \|\!|\nabla \phi \|\!|_{1;[0,T_*]} + \|\!|\mathrm{l.o.t.}\|\!|_{1;[0,T_*]}.\end{aligned}$$

Thus if we choose T_* small enough depending on the constant M, then it follows that

$$\|\!|\nabla \partial_i \partial_j \phi \|\!|_{\beta;[0,T_*]} \lesssim \|\!|\nabla \partial_i \partial_j \phi_* \|\!|_{1;[0,T_*]} + M \|\!|\nabla \phi \|\!|_{1;[0,T_*]} + \|\!|\mathrm{l.o.t.}\|\!|_{1;[0,T_*]}. \tag{3.35}$$

Taking $\beta \to 1$ yields that $\nabla^3 \phi \in \mathrm{C}_1^*[0,T_*]$.

In light of (3.9), $\nabla^2 \omega \in \mathrm{C}_4^*[0,T_*]$. Then by Lemma 2.5, $\nabla^2 v$ has finite $\mathrm{L}^\infty \mathrm{C}^\gamma$-norm on $\mathbb{R}^2 \times [0,T_*]$ for all $\gamma \in (0,1)$. This result and (3.20) show that the right-hand side of (3.34) lies in $\mathrm{L}^\infty\big([0,T_*];\mathrm{C}^\gamma(\mathbb{R}^2)\big)$ for some $\gamma \in (0,1)$. Therefore Lemma 2.1 shows that $\nabla^4\phi - \nabla^4\phi_* \in \mathrm{C}_\beta^*[0,T_*]$ for all $\beta > 1$. Here we used the previous consequence that $\nabla^3 \phi \in \mathrm{C}_1^*[0,T_*]$. Therefore in light of the first estimate in (3.6), $\nabla^4\phi \in \mathrm{C}_\beta^*[0,T_*]$ for all $\beta > 1$. This result and interpolation inequality show that the right-hand side of the last equation in (1.2) has finite $\mathrm{L}^\infty \mathrm{C}^\gamma$-norm for all $\gamma \in (0,1)$, which furthermore shows by (2.4) that $\nabla^2\omega - \nabla^2\omega_* \in \mathrm{C}_\alpha^*[0,T_*]$ for all $\alpha > 2$. Here we have used the fact that $v \cdot \nabla\omega$ and the right-hand side of the last equation in (1.2) lies in $\mathrm{C}_2^*[0,T_*]$. Moreover by the second estimate in (3.6), it holds $\nabla^2\omega_* \in \mathrm{C}_\alpha^*[0,T_*]$ for all $\alpha > 2$. Therefore we can imply from the above arguments that $\nabla^2\omega \in \mathrm{C}_\alpha^*[0,T_*]$ for all $\alpha > 2$. Now we make spatial derivative once for the last equation in (1.2). It turns out that

$$\partial_t \nabla\omega - \Delta \nabla \omega = \mathrm{R}_1 := -\nabla v \cdot \nabla \omega - v \cdot \nabla^2 \omega - \nabla^4 \phi \cdot \nabla \phi - \nabla^3 \phi \cdot \nabla^2 \phi. \tag{3.36}$$

Similar derivation as for (3.35) shows that the $\|\!|\cdot\|\!|_{\alpha;[0,T_*]}$-norm of $\nabla^2\omega$ is uniformly bounded from above by a constant independent of α. Then we take $\alpha \to 2$ and get the optimal exponential decay of $\nabla^2\omega$ at spatial infinity. That is $\nabla^2\omega \in \mathrm{C}_2^*[0,T_*]$.

We are left to show that $\nabla^4\phi \in \mathrm{C}_1^*[0,T_*]$. Since $\nabla^2 v$ has finite $\mathrm{L}^\infty\mathrm{C}^\gamma$-norm on $\mathbb{R}^2 \times [0,T_*]$ for all $\gamma \in (0,1)$ and $\nabla^4\phi \in \mathrm{C}_\beta^*[0,T_*]$ for all $\beta > 1$, the right-hand side of (3.34) has finite $\mathrm{L}^\infty\mathrm{C}^{3/4}$-norm on $\mathbb{R}^2 \times [0,T_*]$. It then follows, by Lemma 2.2, that $\nabla^4\phi - \nabla^4\phi_*$ has finite $\mathrm{L}^\infty\mathrm{C}^{1/2}$-norm on $\mathbb{R}^2 \times [0,T_*]$. Moreover this norm is bounded from above by a constant depending on M. As for $\nabla^4\phi_*$, we do not know that it has finite $\mathrm{L}^\infty\mathrm{C}^{1/2}$-norm on $\mathbb{R}^2 \times [0,T_*]$. But we can represent $\nabla^5\phi_*$ as follows:

$$\nabla^5\phi_*(x,t) = \int_{\mathbb{R}^2} \nabla \mathrm{G}(x-z,t)\, \nabla^4\phi_0(z)\,\mathrm{d}z, \qquad \forall\,(x,t) \in \mathbb{R}^2 \times (0,\infty).$$

Therefore it holds, for all $(x,t) \in \mathbb{R}^2 \times (0,\infty)$, that

$$\big|\nabla^5\phi_*(x,t)\big| \lesssim \int_{\mathbb{R}^2} \mathrm{G}(x-z,t)\,\frac{|x-z|}{t}\,\big|\nabla^4\phi_0(z)\big|\,e^{|z|}\,e^{-|z|} \lesssim \big\|\!\big|\nabla^4\phi_0\big\|\!\big|_1\, t^{-1/2},$$

which furthermore implies the following L^∞-boundedness of $\nabla^5\phi_*$:

$$\big\|\nabla^5\phi_*(\cdot,t)\big\|_\infty \lesssim \big\|\!\big|\nabla^4\phi_0\big\|\!\big|_1\, t^{-1/2}, \qquad \forall\, t \in \big(0,\infty\big). \tag{3.37}$$

By (3.37) and the first estimate in (3.6), we have, with an use of simple interpolation inequality, that

$$\big\|\nabla^4\phi_*(\cdot,t)\big\|_{\mathrm{C}^{1/2}} \lesssim \big\|\nabla^4\phi_*(\cdot,t)\big\|_\infty + \big\|\nabla^5\phi_*(\cdot,t)\big\|_\infty \lesssim_M t^{-1/2}, \qquad \forall\, t \in \big(0,T_*\big].$$

Therefore the above arguments show that

$$\big\|\nabla^4\phi(\cdot,t)\big\|_{\mathrm{C}^{1/2}} \leqslant \big\|\nabla^4\phi(\cdot,t) - \nabla^4\phi_*(\cdot,t)\big\|_{\mathrm{C}^{1/2}} + \big\|\nabla^4\phi_*(\cdot,t)\big\|_{\mathrm{C}^{1/2}} \lesssim_M t^{-1/2}, \qquad \forall\, t \in \big(0,T_*\big]. \tag{3.38}$$

Using the same derivation as for (3.38), by the last eqution in (1.2), we get

$$\big\|\nabla^2\omega(\cdot,t)\big\|_{\mathrm{C}^{1/2}} \lesssim_M t^{-1/2}, \qquad \forall\, t \in \big(0,T_*\big]. \tag{3.39}$$

Now we come back to (3.36). Using ω_* in (3.2) and (2.8), we can represent $\nabla^3\omega - \nabla^3\omega_*$ as follows:

$$\partial_{ij}\nabla\omega(x,t) - \partial_{ij}\nabla\omega_*(x,t) = 2^{-1}\int_0^t\int_{\mathbb{R}^2} \mathrm{G}(x-z,t-s)\big[\mathrm{R}_1(z,s)-\mathrm{R}_1(x,s)\big]\left[\frac{(z_i-x_i)(z_j-x_j)}{2(t-s)^2} - \frac{\delta_{ij}}{t-s}\right]\mathrm{d}z\,\mathrm{d}s.$$

Here (x,t) is a fixed point in $\mathbb{R}^2\times\big(0,T_*\big]$. In light of (3.38)-(3.39) and the fact that $\mathrm{R}_1 \in \mathrm{C}_2^*[0,T_*]$, it holds from the above equality that

$$\begin{aligned}
\big|\partial_{ij}\nabla\omega(x,t) - \partial_{ij}\nabla\omega_*(x,t)\big| \;&\lesssim\; \int_0^t\int_{\mathbb{R}^2} \mathrm{G}(x-z,t-s)\Big|\mathrm{R}_1(z,s)-\mathrm{R}_1(x,s)\Big|\,\frac{\big|z-x\big|^2+(t-s)}{(t-s)^2}\,\mathrm{d}z\,\mathrm{d}s\\
&\lesssim_M \int_0^t\int_{\mathbb{R}^2} \mathrm{G}(x-z,t-s)\Big|\mathrm{R}_1(z,s)-\mathrm{R}_1(x,s)\Big|^{1/2}\Big[e^{-|z|/4}+e^{-|x|/4}\Big]\frac{\big|z-x\big|^2+(t-s)}{(t-s)^2}\,\mathrm{d}z\,\mathrm{d}s\\
&\lesssim_M \int_0^t\int_{\mathbb{R}^2} \mathrm{G}(x-z,t-s)\,\frac{|z-x|^{1/4}}{s^{1/4}}\Big[e^{-|z|/4}+e^{-|x|/4}\Big]\frac{\big|z-x\big|^2+(t-s)}{(t-s)^2}\,\mathrm{d}z\,\mathrm{d}s.
\end{aligned}$$

By the same derivation for (2.5), the above estimate can be reduced to

$$\big|\partial_{ij}\nabla\omega(x,t) - \partial_{ij}\nabla\omega_*(x,t)\big| \;\lesssim_M\; e^{-|x|/4}\,t^{-1/8} \;\lesssim\; e^{-|x|/4}\,t^{-1/2}, \qquad \forall\,(x,t)\in\mathbb{R}^2\times\big(0,T_*\big].$$

Similar derivation as for (3.37) yields that $\big\|\!\big|\nabla^3\omega_*(\cdot,t)\big\|\!\big|_2 \lesssim t^{-1/2}$. Thus it holds by this result and the last estimate that

$$\big\|\!\big|\nabla^3\omega(\cdot,t)\big\|\!\big|_4 \;\lesssim_M\; t^{-1/2}, \qquad \forall\,t\in\big(0,T_*\big].$$

Applying this estimate and Lemma 2.5, we have

$$\big\|\nabla^3 v(\cdot,t)\big\|_\infty \;\lesssim\; \big\|\!\big|\nabla^3\omega(\cdot,t)\big\|\!\big|_4 \;\lesssim_M\; t^{-1/2}, \qquad \forall\,t\in\big(0,T_*\big]. \tag{3.40}$$

Now we make spatial derivative one more time on both sides of (3.34). It follows that

$$\partial_t\big(\nabla^3\phi-\nabla^3\phi_*\big) - \Delta\big(\nabla^3\phi-\nabla^3\phi_*\big) = \mathrm{R}_2 := -\nabla^3 v\cdot\nabla\phi - v\cdot\nabla^4\phi + 2\big(\nabla\phi:\nabla^4\phi\big)\phi + \text{l.o.t.} \tag{3.41}$$

It then turns out by (2.6) that

$$\nabla^4\phi(x,t) - \nabla^4\phi_*(x,t) = 2^{-1}\int_0^t\int_{\mathbb{R}^2}\mathrm{G}(x-z,t-s)\,\mathrm{R}_2(z,s)\,\frac{z-x}{t-s}\,\mathrm{d}z\,\mathrm{d}s.$$

This equality yields, for all $(x,t)\in\mathbb{R}^2\times\big(0,T_*\big]$, that

$$\begin{aligned}
\big|\nabla^4\phi(x,t)-\nabla^4\phi_*(x,t)\big| \;&\lesssim\; M\,e^{-|x|}.+\int_0^t\int_{\mathbb{R}^2}\mathrm{G}(x-z,t-s)\,\big|\nabla^3 v(z,s)\big|\,\big|\nabla\phi(z,s)\big|\,\frac{|z-x|}{t-s}\\
&+\;\Big(\big|v\big|_{0;[0,T_*]}+\big|\nabla\phi\big|_{0;[0,T_*]}\Big)\int_0^t\int_{\mathbb{R}^2}\mathrm{G}(x-z,t-s)\,\big|\nabla^4\phi(z,s)\big|\,\frac{|z-x|}{t-s}.
\end{aligned}$$

The first term on the right-hand side above follows from the term l.o.t. in (3.41). In fact we know that $\text{l.o.t.}\in\mathrm{C}_1^*[0,T_*]$. Therefore (2.3) implies that $\Phi[\text{l.o.t.}]$ also lies in $\mathrm{C}_1^*[0,T_*]$, which gives us the first term on the right-hand side above. Using (3.8), (3.10), (3.40) and the same derivation as for (2.5), we can get from the above estimate that

$$\begin{aligned}
\big|\nabla^4\phi(x,t)-\nabla^4\phi_*(x,t)\big| \;&\lesssim_M\; e^{-|x|}+\int_0^t\int_{\mathbb{R}^2}\mathrm{G}(x-z,t-s)\,s^{-1/2}\,e^{-|z|}\,\frac{|z-x|}{t-s}\\
&+\;\big\|\!\big|\nabla^4\phi\big\|\!\big|_{\beta;[0,T_*]}\int_0^t\int_{\mathbb{R}^2}\mathrm{G}(x-z,t-s)\,e^{-|z|/\beta}\,\frac{|z-x|}{t-s}\\
&\lesssim_M\; e^{-|x|}+e^{-|x|}\int_0^t(t-s)^{-1/2}\,s^{-1/2}\,\mathrm{d}s+\big\|\!\big|\nabla^4\phi\big\|\!\big|_{\beta;[0,T_*]}\,e^{-|x|/\beta}\int_0^t(t-s)^{-1/2}\,\mathrm{d}s\\
&\lesssim_M\; e^{-|x|}+\big\|\!\big|\nabla^4\phi\big\|\!\big|_{\beta;[0,T_*]}\,e^{-|x|/\beta}\,T_*^{1/2}.
\end{aligned}$$

Here β is an arbitrary constant larger than 1. Multiplying $e^{|x|/\beta}$ on both sides of the above estimate and taking supreme over $(x,t) \in \mathbb{R}^2 \times [0, T_*]$, we have

$$\left\| \nabla^4 \phi - \nabla^4 \phi_* \right\|_{\beta;\,[0,T_*]} \leqslant M + M \left\| \nabla^4 \phi \right\|_{\beta;\,[0,T_*]} T_*^{1/2}.$$

Using the first estimate in (3.6), we have $\nabla^4 \phi_* \in \mathrm{C}_1^*\,[0, T_*]$. This result together with the above estimate yield

$$\left\| \nabla^4 \phi \right\|_{\beta;\,[0,T_*]} \leqslant M + M \left\| \nabla^4 \phi \right\|_{\beta;\,[0,T_*]} T_*^{1/2}.$$

Now we choose T_* small enough (smallness depends on M). The last estimate can then be reduced to

$$\left\| \nabla^4 \phi \right\|_{\beta;\,[0,T_*]} \leqslant M.$$

Taking $\beta \to 1$, we know that $\nabla^4 \phi \in \mathrm{C}_1^*\,[0, T_*]$. The proof is then finished. □

We also claim without proof that

Remark 3.3. *Let (ϕ, ω) be the classic solution obtained from Theorem 1.2. $v = \mathrm{K} * \omega$ is the velocity field recovered from ω by the Biot-Savart law. Then for any given $\alpha \in (0,1)$, v has finite $\mathrm{C}^{\alpha/2,\alpha}$-norm on $\mathbb{R}^2 \times [0, T_*]$.*

IV. LOCAL EXISTENCE OF WEAK SOLUTION

In this section we study the local existence of solutions for (1.2) with $\phi_0 \in \mathrm{H}_e^1(\mathbb{R}^2; \mathbb{S}^2)$ and $\omega_0 \in \mathrm{L}^1(\mathbb{R}^2)$. Before we prove Theorem 1.3, two lemmas are given as follows.

Lemma 4.1. *Let (ϕ_0, ω_0) be a smooth initial data on $\mathbb{R}^2$. Moreover we suppose that $(\phi_0 - e, \omega_0)$ is compactly supported on $\mathbb{R}^2$. By Theorem 1.2, for some $T > 0$, the system (1.2) admits a classic solution on $\mathbb{R}^2 \times [0, T]$ with the given initial data (ϕ_0, ω_0). Then for all $p \in (4/3, 2)$ and $t \in [0, T]$, the following estimates hold:*

$$A_p(t) \lesssim_p \max_{s \in [0,t]} s^{1-1/p} \left\| \mathrm{G}(\cdot, s) * \omega_0 \right\|_p + A_p^2(t) + B(t)\, C(t); \tag{4.1}$$

$$B(t) \lesssim_p \max_{s \in [0,t]} s^{1/4} \left\| \mathrm{G}(\cdot, s) * \nabla \phi_0 \right\|_4 + A_p(t)\, B(t) + B^2(t); \tag{4.2}$$

$$C(t) \lesssim_p \max_{s \in [0,t]} s^{1/2} \left\| \nabla \mathrm{G}(\cdot, s) * \nabla \phi_0 \right\|_2 + A_p(t)\, B(t) + A_p(t)\, C(t) + B(t)\, C(t) + B^3(t). \tag{4.3}$$

Here for any $t \in [0, T]$, we define

$$A_p(t) := \max_{s \in [0,t]} s^{1-1/p} \left\| \omega(\cdot, s) \right\|_p, \quad B(t) := \max_{s \in [0,t]} s^{1/4} \left\| \nabla \phi(\cdot, s) \right\|_4, \quad C(t) := \max_{s \in [0,t]} s^{1/2} \left\| \nabla^2 \phi(\cdot, s) \right\|_2. \tag{4.4}$$

Proof. The proof is divided into three steps.

Step 1. Let (x,t) be an arbitrary point in $\mathbb{R}^2 \times [0, T]$. By the last equation in (1.2), $\omega(x,t)$ can be represented as shown below:

$$\omega(x,t) = \int_{\mathbb{R}^2} \mathrm{G}(x - z, t)\, \omega_0(z) + \int_0^t \int_{\mathbb{R}^2} \mathrm{G}(x - z, t - s) \Big[- \nabla_z \cdot \big(\omega v\big) + \nabla_z \cdot \big(\nabla \phi \cdot \Delta \phi\big)^\perp \Big].$$

Integrating by part with respect to the z variable, we get from the above equality that

$$\omega(x,t) = \int_{\mathbb{R}^2} \mathrm{G}(x - z, t)\, \omega_0(z) + \int_0^t \int_{\mathbb{R}^2} \nabla_z \mathrm{G}(x - z, t - s) \cdot \Big[\omega v - \big(\nabla \phi \cdot \Delta \phi\big)^\perp \Big]. \tag{4.5}$$

For any $p \in (4/3, 2)$, $2p/(3p-2)$ and $2p/(4-p)$ are two numbers larger than 1. Thus it holds

$$\begin{aligned} \Big\| \int_0^t \int_{\mathbb{R}^2} \nabla_z \mathrm{G}(x-z, t-s) \cdot \big(\omega v\big) \Big\|_p &\leqslant \int_0^t \Big\| \int_{\mathbb{R}^2} \nabla_z \mathrm{G}(x-z, t-s) \cdot \big(\omega v\big) \Big\|_p \\ &\leqslant \int_0^t \| \nabla \mathrm{G}(\cdot, t-s) \|_{2p/(3p-2)} \| \omega v \|_{2p/(4-p)}. \end{aligned} \tag{4.6}$$

The second inequality in (4.6) follows by Young's inequality for convolutions. With the use of Hölder's inequality and Calderon-Zygmund estimate, ωv can be estimated by

$$\| \omega v \|_{2p/(4-p)} \lesssim \| \omega \|_p \| v \|_{2p/(2-p)} \lesssim_p \| \omega \|_p^2.$$

Applying the last estimate to (4.6) yields

$$\Big\| \int_0^t \int_{\mathbb{R}^2} \nabla_z \mathrm{G}(x-z, t-s) \cdot \big(\omega v\big) \Big\|_p \lesssim_p \int_0^t (t-s)^{-1/p} \| \omega \|_p^2. \tag{4.7}$$

Still using Young's inequality for convolutions and noticing that $4p/(p+4) > 1$, we can show that

$$\begin{aligned} \Big\| \int_0^t \int_{\mathbb{R}^2} \nabla_z \mathrm{G}(x-z, t-s) \cdot \big(\nabla \phi \cdot \Delta \phi\big)^\perp \Big\|_p &\leqslant \int_0^t \Big\| \int_{\mathbb{R}^2} \nabla_z \mathrm{G}(x-z, t-s) \cdot \big(\nabla \phi \cdot \Delta \phi\big)^\perp \Big\|_p \\ &\leqslant \int_0^t \| \nabla \mathrm{G}(\cdot, t-s) \|_{4p/(p+4)} \| \nabla \phi \cdot \Delta \phi \|_{4/3} \\ &\lesssim_p \int_0^t (t-s)^{-5/4+1/p} \| \nabla \phi \cdot \Delta \phi \|_{4/3}. \end{aligned}$$

It then turns out by (4.5), (4.7) and the last estimate that

$$\begin{aligned} \| \omega(\cdot, t) \|_p &\leqslant \| \mathrm{G}(\cdot, t) * \omega_0 \|_p + \mathrm{C}_p \int_0^t (t-s)^{-1/p} \| \omega \|_p^2 + \mathrm{C}_p \int_0^t (t-s)^{-5/4+1/p} \| \nabla \phi \cdot \Delta \phi \|_{4/3} \\ &\lesssim_p \| \mathrm{G}(\cdot, t) * \omega_0 \|_p + \int_0^t (t-s)^{-1/p} \| \omega \|_p^2 + \int_0^t (t-s)^{-5/4+1/p} \| \nabla \phi \|_4 \| \Delta \phi \|_2. \end{aligned} \tag{4.8}$$

Step 2. By the first equation in (1.2), $\partial_j \phi$ can be represented by

$$\partial_j \phi(x, t) = \int_{\mathbb{R}^2} \mathrm{G}(x-z, t) \, \partial_j \phi_0(z) + \int_0^t \int_{\mathbb{R}^2} \partial_{z_j} \mathrm{G}(x-z, t-s) \Big[v \cdot \nabla \phi - |\nabla \phi|^2 \phi \Big]_{(z,s)}. \tag{4.9}$$

Still by Young's inequality for convolution, it can be shown that

$$\begin{aligned} \Big\| \int_0^t \int_{\mathbb{R}^2} \partial_{z_j} \mathrm{G}(x-z, t-s) \big[v \cdot \nabla \phi \big]_{(z,s)} \Big\|_4 &\lesssim \int_0^t \| \nabla \mathrm{G}(\cdot, t-s) \|_{2p/(3p-2)} \| v \cdot \nabla \phi \|_{4p/(4-p)} \\ &\lesssim_p \int_0^t (t-s)^{-1/p} \| v \|_{2p/(2-p)} \| \nabla \phi \|_4 \\ &\lesssim_p \int_0^t (t-s)^{-1/p} \| \omega \|_p \| \nabla \phi \|_4. \end{aligned}$$

The last inequality above used Calderon-Zygmund estimate. Same method can be applied to show that

$$\begin{aligned} \Big\| \int_0^t \int_{\mathbb{R}^2} \partial_{z_j} \mathrm{G}(x-z, t-s) \big[|\nabla \phi|^2 \phi \big]_{(z,s)} \Big\|_4 &\lesssim \int_0^t \| \nabla \mathrm{G}(\cdot, t-s) \|_{4/3} \| |\nabla \phi|^2 \|_2 \\ &\lesssim \int_0^t (t-s)^{-3/4} \| \nabla \phi \|_4^2. \end{aligned}$$

Using the last two estimates, by (4.9), we have

$$\|\nabla\phi(\cdot,t)\|_4 \leqslant \|\mathrm{G}(\cdot,t)*\partial_j\phi_0\|_4 + \mathrm{C}_p\int_0^t (t-s)^{-1/p}\,\|\omega\|_p\,\|\nabla\phi\|_4 + \mathrm{C}\int_0^t (t-s)^{-3/4}\,\|\nabla\phi\|_4^2. \tag{4.10}$$

Taking spatial derivative one more time on both sides of (4.9) implies that

$$\partial_{ij}\phi(x,t) = \int_{\mathbb{R}^2}\partial_i\mathrm{G}(x-z,t)\,\partial_j\phi_0(z) + \int_0^t\int_{\mathbb{R}^2}\partial_{x_i}\mathrm{G}(x-z,t-s)\,\partial_j\Big[v\cdot\nabla\phi - |\nabla\phi|^2\phi\Big]_{(z,s)}. \tag{4.11}$$

By Young's inequality for convolutions, it holds

$$\begin{aligned}\Big\|\int_0^t\int_{\mathbb{R}^2}\partial_{x_i}\mathrm{G}(x-z,t-s)\,\big[\partial_j v\cdot\nabla\phi\big]_{(z,s)}\Big\|_2 &\leqslant \int_0^t\|\nabla\mathrm{G}(\cdot,t-s)\|_{4p/(5p-4)}\,\|\partial_j v\cdot\nabla\phi\|_{4p/(p+4)}\\ &\lesssim_p \int_0^t (t-s)^{-1/4-1/p}\,\|\nabla v\|_p\,\|\nabla\phi\|_4\\ &\lesssim_p \int_0^t (t-s)^{-1/4-1/p}\,\|\omega\|_p\,\|\nabla\phi\|_4.\end{aligned}$$

Similar arguments yield the following three estimates:

$$\begin{aligned}\Big\|\int_0^t\int_{\mathbb{R}^2}\partial_{x_i}\mathrm{G}(x-z,t-s)\,\big[v\cdot\nabla\partial_j\phi\big]_{(z,s)}\Big\|_2 &\leqslant \int_0^t\|\nabla\mathrm{G}(\cdot,t-s)\|_{2p/(3p-2)}\,\|v\cdot\nabla\partial_j\phi\|_p\\ &\lesssim_p \int_0^t (t-s)^{-1/p}\,\|v\|_{2p/(2-p)}\,\|\nabla^2\phi\|_2\\ &\lesssim_p \int_0^t (t-s)^{-1/p}\,\|\omega\|_p\,\|\nabla^2\phi\|_2;\end{aligned}$$

$$\begin{aligned}\Big\|\int_0^t\int_{\mathbb{R}^2}\partial_{x_i}\mathrm{G}(x-z,t-s)\,\big[\nabla\phi:\nabla\partial_j\phi\big]_{(z,s)}\phi(z,s)\Big\|_2 &\leqslant \int_0^t\|\nabla\mathrm{G}(\cdot,t-s)\|_{4/3}\,\|\nabla\phi:\nabla\partial_j\phi\|_{4/3}\\ &\lesssim \int_0^t (t-s)^{-3/4}\,\|\nabla\phi\|_4\,\|\nabla^2\phi\|_2;\end{aligned}$$

$$\begin{aligned}\Big\|\int_0^t\int_{\mathbb{R}^2}\partial_{x_i}\mathrm{G}(x-z,t-s)\,\big[|\nabla\phi|^2\,\partial_j\phi\big]_{(z,s)}\Big\|_2 &\leqslant \int_0^t\|\nabla\mathrm{G}(\cdot,t-s)\|_{4/3}\,\||\nabla\phi|^3\|_{4/3}\\ &\lesssim \int_0^t (t-s)^{-3/4}\,\|\nabla\phi\|_4^3.\end{aligned}$$

Applying the above four estimates to (4.11), we get

$$\begin{aligned}\|\nabla^2\phi(\cdot,t)\|_2 \lesssim_p\ & \|\nabla\mathrm{G}(\cdot,t)*\nabla\phi_0\|_2 + \int_0^t (t-s)^{-1/4-1/p}\,\|\omega\|_p\,\|\nabla\phi\|_4\\ &+ \int_0^t (t-s)^{-1/p}\,\|\omega\|_p\,\|\nabla^2\phi\|_2\\ &+ \int_0^t (t-s)^{-3/4}\,\|\nabla\phi\|_4\,\|\nabla^2\phi\|_2 + \int_0^t (t-s)^{-3/4}\,\|\nabla\phi\|_4^3.\end{aligned} \tag{4.12}$$

Step 3. Recalling the notations defined in (4.4), then by (4.8), we have, for all $s \in [0, t]$, that

$$\begin{aligned} s^{1-1/p} \left\| \omega(\cdot, s) \right\|_p \quad &\lesssim_p \quad s^{1-1/p} \left\| \mathrm{G}(\cdot, s) * \omega_0 \right\|_p + A_p^2(t)\, s^{1-1/p} \int_0^s (s-\tau)^{-1/p}\, \tau^{-2+2/p}\, \mathrm{d}\tau \\ &\qquad + \; B(t)\, C(t)\, s^{1-1/p} \int_0^s (s-\tau)^{-5/4+1/p}\, \tau^{-3/4}\, \mathrm{d}\tau \\ &\lesssim_p \quad \max_{s \in [0,t]} s^{1-1/p} \left\| \mathrm{G}(\cdot, s) * \omega_0 \right\|_p + A_p^2(t) + B(t)\, C(t). \end{aligned}$$

Taking supreme over $s \in [0, t]$ yields (4.1). The proofs for (4.2)-(4.3) are similar. One just needs to use (4.10) and (4.12). □

The estimate (4.1) in Lemma 4.2 also holds when $p = 4/3$. More precisely we have

Lemma 4.2. *Suppose that (ϕ_0, ω_0) and (ϕ, ω) are the same as in Lemma 4.2. Then for all $t \in [0, T]$, the estimate (4.1) also holds if p is taken to be $4/3$.*

Proof. Repeating the same arguments as the derivation for (4.8) yields

$$\left\| \omega(\cdot, t) \right\|_{4/3} \quad \lesssim \quad \left\| \mathrm{G}(\cdot, t) * \omega_0 \right\|_{4/3} + \int_0^t (t-s)^{-3/4} \left\| \omega \right\|_{4/3}^2 + \int_0^t (t-s)^{-1/2} \left\| \nabla \phi \right\|_4 \left\| \Delta \phi \right\|_2.$$

Here t is an arbitrary number in $[0, T]$. Using the same arguments as for (4.10), we have

$$A_{4/3}(t) \quad \lesssim \quad \max_{s \in [0,t]} s^{1/4} \left\| \mathrm{G}(\cdot, s) * \omega_0 \right\|_{4/3} + A_{4/3}^2(t) + B(t)\, C(t), \qquad \forall\, t \in [0, T].$$

The proof is finished. □

Now we prove part (i) of Theorem 1.3.

Proof of (i) in Theorem 1.3. We divide the proof into four steps.

Step 1. Let $(\phi_{0;n}, \omega_{0;n})$ be a sequence of smooth pairs so that as $n \to \infty$,

$$\phi_{0;n} - e \longrightarrow \phi_0 - e, \quad \text{strongly in } \mathrm{H}^1(\mathbb{R}^2); \qquad \omega_{0;n} \longrightarrow \omega_0, \quad \text{strongly in } \mathrm{L}^1(\mathbb{R}^2). \tag{4.13}$$

Here $\phi_{0;n}$ takes values in $\mathbb{S}^2$. Thus for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that

$$\left\| \nabla \phi_{0;m} - \nabla \phi_{0;N} \right\|_2 + \left\| \omega_{0;m} - \omega_{0;N} \right\|_1 \quad \leqslant \quad \epsilon, \qquad \forall\, m > N. \tag{4.14}$$

Moreover we can suppose that $(\phi_{0;n} - e, \omega_{0;n})$ is compactly supported on $\mathbb{R}^2$ for all $n \in \mathbb{N}$. It then turns out, for all $t \in (0, 1)$, that

$$\begin{aligned} t^{1-1/p} \left\| \mathrm{G}(\cdot, t) * \omega_{0;m} \right\|_p \quad &\leqslant \quad t^{1-1/p} \left\| \mathrm{G}(\cdot, t) * \omega_{0;N} \right\|_p + t^{1-1/p} \left\| \mathrm{G}(\cdot, t) * (\omega_{0;m} - \omega_{0;N}) \right\|_p \\ &\lesssim_p \quad t^{1-1/p} \max_{t \in [0,1]} \left\| \mathrm{G}(\cdot, t) * \omega_{0;N} \right\|_p + \left\| \omega_{0;m} - \omega_{0;N} \right\|_1 \\ &\lesssim_p \quad t^{1-1/p} \left\| \left| \mathrm{G}(\cdot, t) * \omega_{0;N} \right| \right\|_{1;[0,1]} + \left\| \omega_{0;m} - \omega_{0;N} \right\|_1 \\ &\lesssim_p \quad t^{1-1/p} \left\| \left| \omega_{0;N} \right| \right\|_1 + \left\| \omega_{0;m} - \omega_{0;N} \right\|_1. \end{aligned}$$

To derive the second inequality above, we used Young's inequality for convolutions. The third inequality is an application of Lemma 2.4 with $\beta = 1$. The last inequality above holds by Lemma 2.3. In light of (4.14) and the last estimate, we can choose $\tau_N > 0$ small enough (the smallness depends on ϵ and $|||\omega_{0;N}|||_1$) so that

$$t^{1-1/p} \left\| \mathrm{G}(\cdot,t) * \omega_{0;m} \right\|_p \quad \lesssim_p \quad \epsilon, \qquad \forall\, t \in [0,\tau_N] \text{ and } m > N. \tag{4.15}$$

Similar arguments can be applied to show that

$$t^{1/4} \left\| \mathrm{G}(\cdot,t) * \nabla\phi_{0;m} \right\|_4 \lesssim \epsilon, \quad t^{1/2} \left\| \nabla \mathrm{G}(\cdot,t) * \nabla\phi_{0;m} \right\|_2 \lesssim \epsilon, \qquad \forall\, t \in [0,\tau_N] \quad \text{and} \quad m > N. \tag{4.16}$$

Here we need (4.14), particularly the bound for the L^2-norm of $\nabla\phi_{0;m} - \nabla\phi_{0;N}$ in (4.14).

Step 2. In the next we fix an $m > N$ and let $p = 8/5$. In light of Theorem 1.2, there exists a $T_m \in [0,\tau_N]$ so that (1.2) admits a classic solution on $\mathbb{R}^2 \times [0,T_m]$ with the given initial data $(\phi_{0;m}, \omega_{0;m})$. Moreover the solution, denoted by (ϕ_m, ω_m), also satisfies

$$(\phi_m, \omega_m) \in \mathrm{C}_1^{*,4}[0,T_m] \times \mathrm{C}_2^{*,2}[0,T_m]. \tag{4.17}$$

Associated with (ϕ_m, ω_m), $A_{m;8/5}(\cdot)$, $B_m(\cdot)$ and $C_m(\cdot)$ are quantities given in (4.4). Here we used a subscript m, which means that these three quantities are defined in terms of (ϕ_m, ω_m). Letting δ be a positive number, we define

$$t_1^* = \sup\left\{ t \in \big(0,T_m\big) : A_{m;8/5}(t) < \delta \right\}, \; t_2^* = \sup\left\{ t \in \big(0,T_m\big) : B_m(t) < \delta \right\}, \; t_3^* = \sup\left\{ t \in \big(0,T_m\big) : C_m(t) < \delta \right\}.$$

Moreover we let s^* be the minimum number between t_1^*, t_2^* and t_3^*. Clearly it satisfies

$$s^* = \min\left\{ t_1^*, \, t_2^*, \, t_3^* \right\} \;\leqslant\; T_m \;\leqslant\; \tau_N.$$

Since $s^* \leqslant t_1^* \wedge t_2^*$, it holds

$$A_{m;8/5}(s^*) \leqslant \delta \qquad \text{and} \qquad B_m(s^*) \leqslant \delta. \tag{4.18}$$

In view of the first estimate in (4.16) and (4.18), (4.2) then yields

$$B_m(s^*) \lesssim \epsilon + \delta\, B_m(s^*).$$

Now we choose δ to be small enough. The above estimate is then reduced to

$$B_m(s^*) \lesssim \epsilon. \tag{4.19}$$

Applying the second estimate in (4.16) and (4.18)-(4.19) to (4.3), we obtain

$$C_m(s^*) \quad \lesssim \quad \epsilon + \delta\, B_m(s^*) + \delta\, C_m(s^*) \quad \lesssim \quad \epsilon + \delta\, C_m(s^*).$$

Therefore we can keep choosing δ small enough so that

$$C_m(s^*) \quad \lesssim \quad \epsilon. \tag{4.20}$$

Similar arguments can be applied to (4.1) and yields

$$A_{m;8/5}(s^*) \quad \lesssim \quad \epsilon. \tag{4.21}$$

Here we need (4.15) and (4.18)-(4.20). In view of (4.19)-(4.21), we can choose ϵ to be small enough so that

$$A_{m;8/5}(s^*) + B_m(s^*) + C_m(s^*) \;\leqslant\; \delta/2. \tag{4.22}$$

By (4.17), ω_m, $\nabla\phi_m$, $\nabla^2\phi_m$ have finite $|||\cdot|||_{2;[0,T_m]}$-norm. Lemma 2.4 then implies that

$$\|\omega_m(\cdot,t)\|_{8/5}, \qquad \|\nabla\phi_m(\cdot,t)\|_4 \quad \text{and} \quad \|\nabla^2\phi_m(\cdot,t)\|_2$$

are continuous functions for $t \in [0, T_m]$. If $s^* < T_m$, then one of t_i^* $(i = 1, 2, 3)$ must be less than T_m. Suppose that $s^* = t_1^* < T_m$ (the cases when $s^* = t_2^*$ and $s^* = t_3^*$ can be similarly treated). Then by the definition of t_1^* at the beginning of this step, we have $A_{m;8/5}(t_1^*) = \delta$. Here we used the continuity of the function $\|\omega(\cdot,t)\|_{8/5}$. On the other hand (4.22) shows that $A_{m;8/5}(t_1^*) = A_{m;8/5}(s^*) \leqslant \delta/2$. This is a contradiction to the fact that $A_{m;8/5}(t_1^*) = \delta$. Thus we have $s^* = T_m$.

Step 3. In this step we extend the existence interval of (ϕ_m, ω_m) from $[0, T_m]$ to $[0, \tau_N]$. Suppose that T_m^* is a number in $[T_m, \tau_N]$ so that (ϕ_m, ω_m) is a classic solution of (1.2) on $\mathbb{R}^2 \times [0, T_m^*)$. Moreover it is assumed to satisfy

$$(\phi_m, \omega_m) \in \mathrm{C}_1^{*,4}[0,T] \times \mathrm{C}_2^{*,2}[0,T], \qquad \text{for all } T < T_m^*. \tag{4.23}$$

Using the same derivation for (4.22), we get

$$A_{m;8/5}(T) + B_m(T) + C_m(T) \;\leqslant\; \delta/2, \qquad \forall\, T < T_m^*.$$

Particularly the above estimate yields

$$T_m^{3/8}\,\big\|\omega_m(\cdot,t)\big\|_{8/5} + T_m^{1/4}\,\big\|\nabla\phi_m(\cdot,t)\big\|_4 + T_m^{1/2}\,\big\|\nabla^2\phi_m(\cdot,t)\big\|_2 \;\lesssim\; 1, \quad \text{for all } t \in \big[T_m/2, T_m^*\big). \tag{4.24}$$

By Calderon-Zygmund estimate, it follows from (4.24) that

$$\big\|v_m(\cdot,t)\big\|_8 \;\lesssim\; \big\|\omega_m(\cdot,t)\big\|_{8/5} \;\lesssim\; T_m^{-3/8}, \quad \text{for all } t \in \big[T_m/2, T_m^*\big). \tag{4.25}$$

Now we consider the equation satisfied by (ϕ_m, v_m). The equation satisfied by ϕ_m can be obtained from the first equation in (1.2). That is

$$\partial_t\phi_m - \Delta\phi_m = -v_m\cdot\nabla\phi_m + \big|\nabla\phi_m\big|^2\phi_m, \qquad \text{on } \mathbb{R}^2 \times [T_m/2, T_m^*). \tag{4.26}$$

Using the second and the last equations in (1.2), we know that v_m satisfies

$$\nabla\times\big(\partial_t v_m - \Delta v_m + v_m\cdot\nabla v_m\big) = -\nabla\times\big(\nabla\phi_m\cdot\Delta\phi_m\big), \qquad \text{on } \mathbb{R}^2 \times [T_m/2, T_m^*).$$

Therefore we can find a p_m so that

$$\partial_t v_m - \Delta v_m + v_m\cdot\nabla v_m = -\nabla p_m - \nabla\phi_m\cdot\Delta\phi_m, \qquad \text{on } \mathbb{R}^2 \times [T_m/2, T_m^*). \tag{4.27}$$

Since $\operatorname{div} v_m = 0$, the equation satisfied by p_m can be derived from the last equation as follows:

$$-\Delta p_m = \operatorname{div}\big(v_m\cdot\nabla v_m\big) + \operatorname{div}\big(\nabla\phi_m\cdot\Delta\phi_m\big). \tag{4.28}$$

Moreover p_m can be represented by

$$p_m(x,t) = -(2\pi)^{-1}\int_{\mathbb{R}^2}\frac{x_j - z_j}{|x-z|^2}\Big[v_m\cdot\nabla v_{m,j} + \partial_j\phi_m\cdot\Delta\phi_m\Big]_{(z,t)}\,\mathrm{d}z.$$

Using this representation and (4.24), by Calderon-Zygmund estimate, we have the following estimate for ∇p_m:

$$\begin{aligned}\big\|\nabla p_m(\cdot,t)\big\|_{4/3} \;&\lesssim\; \big\|v_m(\cdot,t)\cdot\nabla v_m(\cdot,t)\big\|_{4/3} + \big\|\nabla\phi_m(\cdot,t)\cdot\Delta\phi_m(\cdot,t)\big\|_{4/3} \\ &\lesssim\; \big\|v_m(\cdot,t)\big\|_8\,\big\|\nabla v_m(\cdot,t)\big\|_{8/5} + \big\|\nabla\phi_m(\cdot,t)\big\|_4\,\big\|\Delta\phi_m(\cdot,t)\big\|_2 \\ &\lesssim\; \big\|\omega_m(\cdot,t)\big\|_{8/5}^2 + \big\|\nabla\phi_m(\cdot,t)\big\|_4\,\big\|\Delta\phi_m(\cdot,t)\big\|_2 \;\leqslant\; c(T_m), \qquad \forall\, t \in [T_m/2, T_m^*).\end{aligned} \tag{4.29}$$

Here $c(T_m)$ is a constant depending only on T_m. In light that the bounds in (4.24)-(4.25) and (4.29) are independent of $t \in [T_m/2, T_m^*)$, one can apply the standard L^p-estimate for parabolic and elliptic equations (see [18]) to (4.26)-(4.28) and obtain the L^∞-boundedness of $(v_m, \nabla\phi_m)$ on $\mathbb{R}^2 \times [T_m/2, T_m^*)$. Here we also need to use Morrey's inequality. Making derivatives on both sides of (4.26)-(4.28), we can apply similar arguments for the L^∞-boundedness of $(v_m, \nabla\phi_m)$ to get the L^∞-boundedness of the higher-order derivatives of $(v_m, \nabla\phi_m)$ on $\mathbb{R}^2 \times [T_m/2, T_m^*)$. With the L^∞-boundedness obtained above, by Arzelà-Ascoli theorem, (ϕ_m, v_m) and all their higher-order derivatives converge locally uniformly as $t \uparrow T_m^*$. Since $\omega_m = \operatorname{curl} v_m$, we also know that ω_m and all its higher-order derivatives converge locally uniformly as $t \uparrow T_m^*$.

In the remaining of this step, we show the uniform boundedness of $|||\,\phi_m(\cdot,t) - e\,|||_{4;1}$ and $|||\,\omega_m(\cdot,t)\,|||_{2;2}$ for all $t \in [T, T_m^*)$, where T is a number less than T_m^*. Suppose that f is the solution of the following initial value problem:

$$\begin{cases} \partial_t f - \Delta f = 0, & \text{on } \mathbb{R}^2 \times (T, \infty); \\ f(\cdot, T) = \phi_m(\cdot, T). \end{cases}$$

Then by (2.3) in Lemma 2.1, we have, for all $T_1 \in (T, T_m^*)$, that

$$\begin{aligned} |||\,\phi_m - f\,|||_{1;[T,T_1]} + |||\,\nabla\phi_m - \nabla f\,|||_{1;[T,T_1]} \quad &\lesssim \quad (T_1 - T)^{1/2} \left|\left|\left| v_m \cdot \nabla\phi_m + \left|\nabla\phi_m\right|^2 \phi_m \right|\right|\right|_{1;[T,T_1]} \\ &\lesssim_{c_*} \quad \left(T_m^* - T\right)^{1/2} |||\,\nabla\phi_m\,|||_{1;[T,T_1]}, \end{aligned}$$

where c_* is a constant depending on the L^∞-norm of $(v_m, \nabla\phi_m)$ on $\mathbb{R}^2 \times \left[T, T_m^*\right)$. Employing Lemma 2.3 yields

$$|||\,f - e\,|||_{1;[T,T_1]} + |||\,\nabla f\,|||_{1;[T,T_1]} \quad \lesssim \quad |||\,\phi_m(\cdot,T) - e\,|||_1 + |||\,\nabla\phi_m(\cdot,T)\,|||_1.$$

It then turns out by the last two estimates that

$$|||\,\phi_m - e\,|||_{1;[T,T_1]} + |||\,\nabla\phi_m\,|||_{1;[T,T_1]} \quad \lesssim \quad |||\,\phi_m(\cdot,T) - e\,|||_{1;1} + c_* \left(T_m^* - T\right)^{1/2} |||\,\nabla\phi_m\,|||_{1;[T,T_1]}.$$

Now we choose T so that $T_m^* - T$ is sufficiently small (smallness depends on the constant c_*). The above estimate can then be reduced to

$$|||\,\phi_m - e\,|||_{1;1;[T,T_1]} \quad \lesssim \quad |||\,\phi_m(\cdot,T) - e\,|||_{1;1}.$$

Therefore $|||\cdot|||_{1;1}$-norm of $\phi_m(\cdot,t) - e$ is uniformly bounded for all $t \in [T, T_m^*)$. This shows that the limit of $\phi_m - e$ as $t \uparrow T_m^*$ has finite $|||\cdot|||_{1;1}$-norm. In light of (4.23), we can repeat the method used above and show that the limit of $(\phi_m - e, \omega_m)$ as $t \uparrow T_m^*$ is contained in the space $\mathrm{C}_1^{*,4}(\mathbb{R}^2) \times \mathrm{C}_2^{*,2}(\mathbb{R}^2)$. Letting the limit of (ϕ_m, ω_m) as $t \uparrow T_m^*$ be an initial data at T_m^*, by Theorem 1.2, we can keep solving the equation (1.2) to a time inverval $[T_m^*, T_m^* + \epsilon)$. By this way we can extend the solution (ϕ_m, ω_m) till the time arrives at τ_N.

Step 4. In the last step we have shown that the solution (ϕ_m, ω_m) can be extended to the time interval $[0, \tau_N]$ for all $m > N$. Using the same method as for (4.22), we know that

$$A_{m;8/5}(\tau_N) + B_m(\tau_N) + C_m(\tau_N) \ \leqslant \ \delta/2.$$

Thus for all $\tau \in (0, \tau_N)$, it holds

$$\tau^{3/8} \left\|\omega_m(\cdot,t)\right\|_{8/5} + \tau^{1/4} \left\|\nabla\phi_m(\cdot,t)\right\|_4 + \tau^{1/2} \left\|\nabla^2\phi_m(\cdot,t)\right\|_2 \ \lesssim \ 1, \quad \text{for all } t \in \left[\tau, \tau_N\right]. \tag{4.30}$$

By Calderon-Zygmund estimate, it follows from the above estimate that

$$\left\|v_m(\cdot,t)\right\|_8 \ \lesssim \ \left\|\omega_m(\cdot,t)\right\|_{8/5} \ \lesssim \ \tau^{-3/8}, \quad \text{for all } t \in \left[\tau, \tau_N\right].$$

Same arguments as for (4.29) yields

$$\| \nabla p_m(\cdot,t) \|_{4/3} \lesssim \| \omega_m(\cdot,t) \|_{8/5}^2 + \| \nabla \phi_m(\cdot,t) \|_4 \| \Delta \phi_m(\cdot,t) \|_2 \leqslant c(\tau), \qquad \forall\, t \in [\tau, \tau_N].$$

Here $c(\tau)$ is a constant depending only on τ. In light of the last three estimates, by the same arguments as in Step 3, we know that (ϕ_m, v_m, ω_m) (also their higher-order derivatives) are L^∞-bounded in $\mathbb{R}^2 \times [\tau, \tau_N]$. Moreover the upper bound is independent of m. Therefore by Arzelà-Ascoli theorem and a diagonal process, we can extract a subsequence, still denoted by (ϕ_m, v_m, ω_m) so that as $m \to \infty$, this sequence converges locally uniformly on $\mathbb{R}^2 \times (0, \tau_N]$. Now we denote by (ϕ, v, ω) the limit of (ϕ_m, v_m, ω_m) as $m \to \infty$. Clearly on $\mathbb{R}^2 \times (0, \tau_N)$, it solves the first and third equations in (1.2) smoothly. Now we show that

$$v = \mathrm{K} * \omega. \tag{4.31}$$

In view of (4.30), for any $t \in [\tau, \tau_N]$, $\omega_m(\cdot,t)$ converges weakly in $\mathrm{L}^{8/5}$ to $\omega(\cdot,t)$. Letting ψ be a smooth test function compactly supported on $\mathbb{R}^2$, then we have, for all $t \in [\tau, \tau_N]$, that

$$\int_{\mathbb{R}^2} \psi(x)\, v_m(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^2} \psi(x)\,\mathrm{d}x \int_{\mathbb{R}^2} \mathrm{K}(x-z)\,\omega_m(z,t)\,\mathrm{d}z = \int_{\mathbb{R}^2} \omega_m(z,t)\,\mathrm{d}z \int_{\mathbb{R}^2} \mathrm{K}(x-z)\,\psi(x)\,\mathrm{d}x. \tag{4.32}$$

Since v_m converges locally uniformly to v on $\mathbb{R}^2 \times [\tau, \tau_N]$, the most-left-hand side of (4.32) satisfies

$$\int_{\mathbb{R}^2} \psi(x)\, v_m(x,t)\,\mathrm{d}x \longrightarrow \int_{\mathbb{R}^2} \psi(x)\, v(x,t)\,\mathrm{d}x, \qquad \text{as } m \to \infty.$$

In light that $\mathrm{K} * \psi \in \mathrm{L}^{8/3}$, applying the $\mathrm{L}^{8/5}$-weak convergence of $\omega_m(\cdot,t)$ then yields

$$\int_{\mathbb{R}^2} \omega_m(z,t)\,\mathrm{d}z \int_{\mathbb{R}^2} \mathrm{K}(x-z)\,\psi(x)\,\mathrm{d}x \longrightarrow \int_{\mathbb{R}^2} \omega(z,t)\,\mathrm{d}z \int_{\mathbb{R}^2} \mathrm{K}(x-z)\,\psi(x)\,\mathrm{d}x, \qquad \text{as } m \to \infty.$$

Employing the last two convergence, we then can take $m \to \infty$ in (4.32) and obtain

$$\int_{\mathbb{R}^2} \psi(x)\, v(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^2} \omega(z,t)\,\mathrm{d}z \int_{\mathbb{R}^2} \mathrm{K}(x-z)\,\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^2} \psi(x)\,\mathrm{d}x \int_{\mathbb{R}^2} \mathrm{K}(x-z)\,\omega(z,t)\,\mathrm{d}z.$$

(4.31) then follows.

In the remaining of the proof, we only need show that $(\phi(\cdot,t), v(\cdot,t), \omega(\cdot,t))$ converges to (ϕ_0, v_0, ω_0) as $t \downarrow 0$, in the sense given in Theorem 1.3. Let (ϕ_m, v_m, ω_m) be the convergent subsequence obtained above. Using the same derivations as for (4.19)-(4.21), for any $\epsilon > 0$, we can find a $N' > N$ and $\tau_{N'} < \tau_N$ so that

$$A_{m;\,4/3}(\tau_{N'}) + B_m(\tau_{N'}) + C_m(\tau_{N'}) \leqslant \epsilon, \qquad \forall\, m > N'. \tag{4.33}$$

Here we used Lemma 4.2 so that the estimate for $A_{m;p}$ in (4.1) is valid when $p = 4/3$. Using Calderon-Zygmund estimate and the estimate for $A_{m;\,4/3}(\tau_{N'})$ in (4.33), we can bound the L^4-norm of v_m as follows:

$$\| v_m(\cdot,t) \|_4 \lesssim \| \omega_m(\cdot,t) \|_{4/3} \leqslant \epsilon\, t^{-1/4}, \qquad \forall\, t \in (0, \tau_{N'}]. \tag{4.34}$$

Now we prove the L^2-convergence of $\phi(\cdot,t) - e$ as $t \downarrow 0$. In light of (4.26), $\phi_m - e$ can be represented by

$$\phi_m(x,t) - e = \mathrm{G}(\cdot,t) * (\phi_{0;\,m} - e) + \int_0^t \int_{\mathbb{R}^2} \mathrm{G}(x-z, t-s) \left[- v_m \cdot \nabla \phi_m + \left| \nabla \phi_m \right|^2 \phi_m \right]_{(z,s)} \mathrm{d}z\,\mathrm{d}s. \tag{4.35}$$

Here (x,t) is an arbitrary point in $\mathbb{R}^2 \times (0, \tau_{N'})$. It then turns out, by the above equality, that

$$\begin{aligned} \left\| \left(\phi_m(\cdot,t) - e \right) - \mathrm{G}(\cdot,t) * (\phi_{0;\,m} - e) \right\|_2 &\leqslant \int_0^t \left\| - v_m(\cdot,s) \cdot \nabla \phi_m(\cdot,s) + \left| \nabla \phi_m \right|^2(\cdot,s)\, \phi_m(\cdot,s) \right\|_2 \mathrm{d}s \\ &\lesssim \int_0^t \| v_m(\cdot,s) \|_4 \| \nabla \phi_m(\cdot,s) \|_4 + \| \nabla \phi_m(\cdot,s) \|_4^2 \,\mathrm{d}s. \end{aligned}$$

With (4.33)-(4.34), this estimate can be reduced to

$$\big\| \big(\phi_m(\cdot,t)-e\big) - \mathrm{G}(\cdot,t) * (\phi_{0;m}-e) \big\|_2 \;\lesssim\; \epsilon^2\, t^{1/2}.$$

Since $\big(\phi_m(\cdot,t)-e\big) - \mathrm{G}(\cdot,t) * (\phi_{0;m}-e)$ converges to $\big(\phi(\cdot,t)-e\big) - \mathrm{G}(\cdot,t) * (\phi_0-e)$ pointwisely as $m\to\infty$, by Fatou's lemma, we can take $m\to\infty$ in the above estimate and get

$$\big\| \big(\phi(\cdot,t)-e\big) - \mathrm{G}(\cdot,t) * (\phi_0-e) \big\|_2 \;\lesssim\; \epsilon^2\, t^{1/2}, \qquad \forall\, t\in(0,\tau_{N'}).$$

This estimate shows that $\phi - e$ has finite $\mathrm{L}^\infty_t\mathrm{L}^2_x$-norm on $\mathbb{R}^2\times(0,\tau_{N'})$. In light that $\mathrm{G}(\cdot,t)*(\phi_0-e)$ converges to $\phi_0 - e$ strongly in L^2 as $t\downarrow 0$, the above estimate also implies that $\phi(\cdot,t)-e$ converges to ϕ_0-e strongly in L^2 as $t\downarrow 0$. Taking spatial derivative once on both sides of (4.35), we get

$$\nabla\phi_m(x,t) = \mathrm{G}(\cdot,t)*\nabla\phi_{0;m} + \int_0^t\int_{\mathbb{R}^2} \nabla\mathrm{G}(x-z,t-s)\big[-v_m\cdot\nabla\phi_m + \big|\nabla\phi_m\big|^2\phi_m\big]_{(z,s)}\,\mathrm{d}z\,\mathrm{d}s.$$

Thus it holds for all $t\in[0,\tau_{N'}]$ that

$$\begin{aligned}
\big\|\nabla\phi_m(\cdot,t) - \mathrm{G}(\cdot,t)*\nabla\phi_{0;m}\big\|_2 \;&\leqslant\; \int_0^t (t-s)^{-1/2}\,\big\| -v_m(\cdot,s)\cdot\nabla\phi_m(\cdot,s) + \big|\nabla\phi_m\big|^2(\cdot,s)\,\phi_m(\cdot,s)\big\|_2\,\mathrm{d}s\\
&\lesssim\; \int_0^t (t-s)^{-1/2}\Big(\big\|v_m(\cdot,s)\big\|_4\big\|\nabla\phi_m(\cdot,s)\big\|_4 + \big\|\nabla\phi_m(\cdot,s)\big\|_4^2\Big)\,\mathrm{d}s\\
&\lesssim\; \epsilon^2\int_0^t (t-s)^{-1/2}\,s^{-1/2}\,\mathrm{d}s \;\lesssim\; \epsilon^2.
\end{aligned}\tag{4.36}$$

Here we also used (4.33)-(4.34). Still by Fatou's lemma, we can take $m\to\infty$ in the above estimate and get

$$\big\|\nabla\phi(\cdot,t) - \mathrm{G}(\cdot,t)*\nabla\phi_0\big\|_2 \;\lesssim\; \epsilon^2, \qquad \forall\; t\in(0,\tau_{N'}). \tag{4.37}$$

This estimate shows that $\nabla\phi$ has finite $\mathrm{L}^\infty_t\mathrm{L}^2_x$-norm on $\mathbb{R}^2\times(0,\tau_{N'})$. Since $\mathrm{G}(\cdot,t)*\nabla\phi_0$ converges to $\nabla\phi_0$ strongly in L^2 as $t\downarrow 0$, (4.37) also implies that $\nabla\phi(\cdot,t)$ converges to $\nabla\phi_0$ strongly in L^2 as $t\downarrow 0$. Similar arguments can be applied to the vorticity ω. By the last equation in (1.2), ω_m can be represented by

$$\omega_m(x,t) = \int_{\mathbb{R}^2}\mathrm{G}(x-z,t)\,\omega_{0;m}(z) + \int_0^t\int_{\mathbb{R}^2}\nabla_z\mathrm{G}(x-z,t-s)\cdot\Big[\omega_m v_m - \big(\nabla\phi_m\cdot\Delta\phi_m\big)^\perp\Big]_{(z,s)}\,\mathrm{d}z\,\mathrm{d}s.$$

In view of Young's inequality for convolutions, it then follows from the above estimate that

$$\begin{aligned}
\big\|\omega_m(\cdot,t) - \mathrm{G}(\cdot,t)*\omega_{0;m}\big\|_1 \;&\lesssim\; \int_0^t \big\|\nabla\mathrm{G}(\cdot,t-s)\big\|_1\big\|\omega_m(\cdot,s)\,v_m(\cdot,s)\big\|_1\\
&\quad+\; \int_0^t \big\|\nabla\mathrm{G}(\cdot,t-s)\big\|_1\big\|\nabla\phi_m(\cdot,s)\cdot\Delta\phi_m(\cdot,s)\big\|_1.
\end{aligned}$$

By Hölder's inequality and Calderon-Zygmund estimate, the last estimate yields

$$\begin{aligned}
\big\|\omega_m(\cdot,t) - \mathrm{G}(\cdot,t)*\omega_{0;m}\big\|_1 \;&\lesssim\; \int_0^t (t-s)^{-1/2}\big\|\omega_m(\cdot,s)\,v_m(\cdot,s)\big\|_1 + \int_0^t (t-s)^{-1/2}\big\|\nabla\phi_m(\cdot,s)\cdot\Delta\phi_m(\cdot,s)\big\|_1\\
&\lesssim\; \int_0^t (t-s)^{-1/2}\big\|\omega_m(\cdot,s)\big\|_{4/3}\big\|v_m(\cdot,s)\big\|_4 + \int_0^t (t-s)^{-1/2}\big\|\nabla\phi_m(\cdot,s)\big\|_2\big\|\Delta\phi_m(\cdot,s)\big\|_2\\
&\lesssim\; \int_0^t (t-s)^{-1/2}\big\|\omega_m(\cdot,s)\big\|_{4/3}^2 + \max_{s\in(0,\tau_{N'})}\big\|\nabla\phi_m(\cdot,s)\big\|_2\int_0^t (t-s)^{-1/2}\big\|\Delta\phi_m(\cdot,s)\big\|_2.
\end{aligned}$$

In light of (4.36), $\| \nabla \phi_m(\cdot, s) \|_2$ is uniformly bounded for all $s \in (0, \tau_{N'})$. The upper bound depends only on the L^2-norm of $\nabla \phi_0$. Using result together with (4.33), we can reduce the last estimate to

$$\| \omega_m(\cdot, t) - \mathrm{G}(\cdot, t) * \omega_{0;m} \|_1 \lesssim \epsilon^2 \int_0^t (t-s)^{-1/2} s^{-1/2} \tag{4.38}$$

$$+ \; c\big(\| \nabla \phi_0 \|_2 \big)\, \epsilon \int_0^t (t-s)^{-1/2} s^{-1/2} \leqslant c\big(\| \nabla \phi_0 \|_2 \big)\, \epsilon, \quad \forall\, t \in (0, \tau_{N'}).$$

Here $c\big(\| \nabla \phi_0 \|_2 \big)$ is a constant depending on the L^2-norm of $\nabla \phi_0$. Still by Fatou's lemma, we can take $m \to \infty$ in the above estimate and get

$$\| \omega(\cdot, t) - \mathrm{G}(\cdot, t) * \omega_0 \|_1 \leqslant c\big(\| \nabla \phi_0 \|_2 \big)\, \epsilon, \qquad \forall\, t \in \big(0, \tau_{N'}\big). \tag{4.39}$$

This estimate shows that ω has finite $\mathrm{L}^\infty \mathrm{L}^1$-norm on $\mathbb{R}^2 \times (0, \tau_{N'})$. Since $\mathrm{G}(\cdot, t) * \omega_0$ converges to ω_0 strongly in L^1 as $t \downarrow 0$, (4.39) also implies that $\omega(\cdot, t)$ converges to ω_0 strongly in L^1 as $t \downarrow 0$. The convergence of $v(\cdot, t)$ follows by a simple duality argument. In fact for any $\psi \in \mathrm{C}_c^\infty(\mathbb{R}^2)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^2} \psi(x) \big(v(x,t) - v_0(x) \big)\, \mathrm{d}x &= \int_{\mathbb{R}^2} \psi(x) \int_{\mathbb{R}^2} \mathrm{K}(x-z) \big(\omega(z,t) - \omega_0(z) \big)\, \mathrm{d}z\, \mathrm{d}x \\
&= \int_{\mathbb{R}^2} \big(\omega(z,t) - \omega_0(z) \big) \int_{\mathbb{R}^2} \mathrm{K}(x-z)\, \psi(x)\, \mathrm{d}x\, \mathrm{d}z \\
&= - \int_{\mathbb{R}^2} \big(\omega(z,t) - \omega_0(z) \big) \int_{\mathbb{R}^2} \mathrm{K}(z-x)\, \psi(x)\, \mathrm{d}x\, \mathrm{d}z.
\end{aligned}$$

Thus it holds, for all $p > 2$, that

$$\left| \int_{\mathbb{R}^2} \psi(x) \big(v(x,t) - v_0(x) \big)\, \mathrm{d}x \right| \leqslant \| \omega(\cdot, t) - \omega_0(\cdot) \|_1 \, \| \mathrm{K} * \psi \|_\infty \lesssim_p \| \omega(\cdot, t) - \omega_0(\cdot) \|_1 \Big(\| \psi \|_1 + \| \psi \|_p \Big).$$

By density arguments, the above estimate still holds for all $\psi \in \mathrm{L}^1 \cap \mathrm{L}^p$. Taking supreme over all $\psi \in \mathrm{L}^1 \cap \mathrm{L}^p$, we get from the above estimate that

$$\| v(\cdot, t) - v_0(\cdot) \|_{(\mathrm{L}^1 \cap \mathrm{L}^p)^*} \lesssim_p \| \omega(\cdot, t) - \omega_0(\cdot) \|_1.$$

The convergence of $v(\cdot, t)$ then follows since $\omega(\cdot, t) \to \omega_0$ strongly in L^1 as $t \downarrow 0$. □

Slight modifications of the above proof leads to

Remark 4.3. *Let (ϕ_0, ω_0) be the same as in Theorem 1.3. Then we can extend the solution obtained in Theorem 1.3 to a global solution defined on $\mathbb{R}^2 \times (0, \infty)$, provided that $\| \nabla \phi_0 \|_2 + \| \omega_0 \|_1 \leqslant \epsilon$. Here $\epsilon > 0$ is a number suitably small. Moreover on $\mathbb{R}^2 \times (0, \infty)$, the extended solution is smooth.*

V. GLOBAL EXISTENCE OF WEAK SOLUTION

This section is devoted to finishing the proofs of Theorems 1.3-1.4. The key point to extend a solution globally in time is a global energy inequality concerning the kinetic energy of v and the L^2-norm of $\nabla \phi$. However formally from the last equation in (1.2),

$$\Omega(t) := \int_{\mathbb{R}^2} \omega(x,t)\, \mathrm{d}x$$

is a conserved quantity. If initially $\Omega(0)$ does not equal to 0, then for all $t > 0$, $\Omega(t)$ should not be 0. In light of Proposition 3.3 in [5], we can not expect that the kinetic energy of v is finite. A decomposition of v is required.

Still by Proposition 3.3 in [5], if we want some part of the velocity v has finite kinetic energy, then the vorticity function obtained by taking curl of this part should have zero value when it is integrated over $\mathbb{R}^2$. Upon this consideration, we decompose v into the sum given in (1.4). Formally by (1.3) and the last equation in (1.2), the curl of v^* is a conserved quantity and satisfies

$$\int_{\mathbb{R}^2\times\{t\}} \operatorname{curl} v^* = \int_{\mathbb{R}^2\times\{t\}} \omega - \int_{\mathbb{R}^2\times\{t\}} \bar{\omega} = \int_{\mathbb{R}^2\times\{\tau\}} \omega - \int_{\mathbb{R}^2\times\{\tau\}} \bar{\omega} = 0, \qquad \text{for all } t \in [\tau, T_*].$$

Thus we can expect that the kinetic energy of v^* is finite. This is exactly part (ii) of Theorem 1.3. Before proving it, in the next, we give a global energy inequality. That is

Lemma 5.1 (**Global Energy Inequality**). *Given $t_1 < t_2$, we suppose that $(\psi, \bar{u}, u^*)$ is a weak solution of the following system:*

$$\begin{cases} \partial_t \psi + u^* \cdot \nabla \psi - \Delta \psi = -\bar{u} \cdot \nabla \psi + |\nabla \psi|^2 \psi, & \text{on } \mathbb{R}^2 \times (t_1, t_2); \\ \partial_t u^* + u^* \cdot \nabla u^* - \Delta u^* = -u^* \cdot \nabla \bar{u} - \bar{u} \cdot \nabla u^* - \nabla p^* - \nabla \cdot \big(\nabla \psi \odot \nabla \psi\big), & \text{on } \mathbb{R}^2 \times \big(t_1, t_2\big); \\ \operatorname{div} \bar{u} = \operatorname{div} u^* = 0. \end{cases} \tag{5.1}$$

Here p^* is a pressure. ψ is an $\mathbb{S}^2$-valued map. If in addition we have

$$\begin{cases} \nabla \psi \in \mathrm{L}^\infty\big([t_1, t_2]; \mathrm{L}^2\big) \cap \mathrm{L}^2\big([t_1, t_2]; \mathrm{H}^1\big); \\ p^* \in \mathrm{L}^{4/3}\big([t_1, t_2]; \mathrm{W}^{1,4/3}\big); \\ \bar{u} \in \mathrm{L}^1\big([t_1, t_2]; \mathrm{W}^{1,\infty}\big) \quad \text{and} \quad u^* \in \mathrm{L}^\infty\big([t_1, t_2]; \mathrm{L}^2\big) \cap \mathrm{L}^2\big([t_1, t_2]; \mathrm{H}^1\big), \end{cases} \tag{5.2}$$

then it holds

$$\int_{\mathbb{R}^2\times\{t_2\}} \big|u^*\big|^2 + \big|\nabla \psi\big|^2 + \int_{t_1}^{t_2}\int_{\mathbb{R}^2} \big|\nabla u^*\big|^2 + \big|\Delta \psi + |\nabla \psi|^2 \psi\big|^2 \leqslant \exp\left\{ \mathrm{c} \int_{t_1}^{t_2} \big\|\nabla \bar{u}\big\|_\infty \right\} \int_{\mathbb{R}^2\times\{t_1\}} \big|u^*\big|^2 + \big|\nabla \psi\big|^2.$$

Here $\mathrm{c} > 0$ is an universal constant.

The proof of Lemma 5.1 follows similarly as the proof of Lemma 4.1 in [20]. We omit it here. The following lemma is also required in the proof of Theorem 1.3, which is an improvement of Proposition 3.3 in [5].

Lemma 5.2. *Suppose that $w \in \mathrm{C}^*_\beta\big(\mathbb{R}^2\big)$ for some $\beta > 0$. Then $u = \mathrm{K} * w \in \mathrm{L}^2(\mathbb{R}^2)$ if and only if*

$$\int_{\mathbb{R}^2} w = 0. \tag{5.3}$$

If the $\||\cdot\||_\beta$-norm of w is bounded from above by a constant W, then the $\mathrm{L}^2(\mathbb{R}^2)$-norm of u is bounded from above by a constant depending on W.

Proof: In light of Lemma 2.5, u is uniformly bounded on $\mathbb{R}^2$. The upper bound depends on the $\||\cdot\||_\beta$-norm of w. Therefore we only need to study the L^2-integrability of u on B^c_R. Here R is a positive radius sufficiently large. For any $x \in \mathrm{B}^c_\mathrm{R}$, $u(x)$ can be rewritten as follows:

$$\begin{aligned} u(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-z)^\perp}{|x-z|^2} w(z)\,\mathrm{d}z \\ &= \frac{1}{2\pi} \frac{x^\perp}{|x|^2} \int_{\mathbb{R}^2} w(z)\,\mathrm{d}z + \frac{1}{2\pi} \int_{\mathbb{R}^2} \left[\frac{(x-z)^\perp}{|x-z|^2} - \frac{x^\perp}{|x|^2} \right] w(z)\,\mathrm{d}z. \end{aligned} \tag{5.4}$$

The last term in (5.4) can be further written as

$$\begin{aligned} \int_{\mathbb{R}^2} \left[\frac{(x-z)^\perp}{|x-z|^2} - \frac{x^\perp}{|x|^2} \right] w(z)\,\mathrm{d}z &= \int_{\left\{|z|<|x|^{1/2}\right\}} \left[\frac{(x-z)^\perp}{|x-z|^2} - \frac{x^\perp}{|x|^2} \right] w(z)\,\mathrm{d}z \\ &\quad + \int_{\left\{|z|\geqslant|x|^{1/2}\right\}} \left[\frac{(x-z)^\perp}{|x-z|^2} - \frac{x^\perp}{|x|^2} \right] w(z)\,\mathrm{d}z. \end{aligned} \tag{5.5}$$

Since $|x| > \mathrm{R}$ with R sufficiently large, then for all z with $|z| < |x|^{1/2}$, we have

$$|x-z|^{-2} = |x|^{-2}\left(1 - \frac{2x\cdot z}{|x|^2} + \frac{|z|^2}{|x|^2}\right)^{-1} = |x|^{-2} + \mathcal{O}\big(|x|^{-5/2}\big).$$

It turns out that

$$\left|\frac{(x-z)^\perp}{|x-z|^2} - \frac{x^\perp}{|x|^2}\right| \lesssim |x|^{-3/2}, \qquad \text{for all } x \text{ and } z \text{ satisfying } |x| > \mathrm{R} \text{ and } |z| < |x|^{1/2}.$$

Applying the above estimate to the first term on the right-hand side of (5.5), we get

$$\left|\int_{\{|z|<|x|^{1/2}\}}\left[\frac{(x-z)^\perp}{|x-z|^2} - \frac{x^\perp}{|x|^2}\right] w(z)\,\mathrm{d}z\right| \lesssim |x|^{-3/2}\int_{\mathbb{R}^2}|w| \leqslant |x|^{-3/2}\,|||\,w\,|||_\beta \int_{\mathbb{R}^2} e^{-|z|/\beta}\,\mathrm{d}z. \tag{5.6}$$

If $|z| \geqslant |x|^{1/2}$, then $w(z)$ satisfies

$$|w(z)| \leqslant |||\,w\,|||_\beta\, e^{-|z|/\beta} \leqslant |||\,w\,|||_\beta\, e^{-|z|/2\beta}\, e^{-|x|^{1/2}/2\beta}.$$

Thus the second term on the right-hand side of (5.5) can be estimated by

$$\begin{aligned}\left|\int_{\{|z|\geqslant|x|^{1/2}\}}\left[\frac{(x-z)^\perp}{|x-z|^2} - \frac{x^\perp}{|x|^2}\right] w(z)\,\mathrm{d}z\right| &\lesssim_{|||\,w\,|||_\beta} |x|^{-1}\, e^{-|x|^{1/2}/2\beta}\int_{\mathbb{R}^2} e^{-|z|/2\beta}\,\mathrm{d}z \\ &\quad + e^{-|x|^{1/2}/2\beta}\int_{\mathbb{R}^2}\frac{1}{|x-z|}\, e^{-|z|/2\beta}\,\mathrm{d}z \\ &\lesssim_{\beta,\,|||\,w\,|||_\beta} e^{-|x|^{1/2}/2\beta}.\end{aligned} \tag{5.7}$$

In view of (5.6)-(5.7), the last term in (5.4) is L^2-integrable on B_R^c. Therefore the L^2-integrability of u on B_R^c is equivalent to the L^2-integrability of the first term on the second line of (5.4), which is L^2-integrable on B_R^c if and only if (5.3) holds. The proof is finished. □

In the next, we prove part (ii) and (iii) of Theorem 1.3.

Proof of (ii) and (iii) in Theorem 1.3. The arguments in the following are continued from the last section, where part (i) of Theorem 1.3 was proved.

Step 5. In the proof of part (i) of Theorem 1.3 (see Step 4 there), the approximation solutions (ϕ_m, ω_m) were shown to exist on the time interval $(0, \tau_{N'}]$, for all $m > N'$. Moreover for a fixed $\tau \in (0, \tau_{N'})$, the L^∞-norm of ω_m on $\mathbb{R}^2 \times [\tau, \tau_{N'}]$ are uniformly bounded from above by a constant independent of m. Thus by (4.38), it follows that

$$\|\omega_m(\cdot,t)\|_p \leqslant c_1, \qquad \forall\, m > N',\ p \in [1,\infty] \text{ and } t \in [\tau, \tau_{N'}]. \tag{5.8}$$

Here c_1 is a positive constant depending on p, τ and the initial data (ϕ_0, ω_0). In view of (4.33), it holds

$$\|\nabla^2\phi_m(\cdot,t)\|_2 \leqslant c(\tau), \qquad \forall\, m > N' \text{ and } t \in [\tau, \tau_{N'}].$$

With this estimate and (4.36), it follows that

$$\|\nabla\phi_m(\cdot,t)\|_2 + \|\nabla^2\phi_m(\cdot,t)\|_2 \leqslant c_2, \qquad \forall\, m > N' \text{ and } t \in [\tau, \tau_{N'}]. \tag{5.9}$$

Here c_2 is a constant depending on τ and the L^2-norm of $\nabla\phi_0$. Taking $m \to \infty$ in (5.8)-(5.9), by Fatou's lemma, we have

$$\|\omega(\cdot,t)\|_p + \|\nabla\phi(\cdot,t)\|_2 + \|\nabla^2\phi(\cdot,t)\|_2 \leqslant c_1 + c_2, \qquad \forall\, p \in [1,\infty] \text{ and } t \in [\tau, \tau_{N'}]. \tag{5.10}$$

In the next we consider the velocity field $v_m = \mathrm{K}*\omega_m$. Let $(\bar\omega_m, \bar v_m)$ be the unique mild solution of the following initial value problem:

$$\begin{cases} \partial_t \bar\omega_m - \Delta\bar\omega_m + \bar v_m \cdot \nabla\bar\omega_m = 0, & \text{on } \mathbb{R}^2\times(\tau,\infty); \\ \bar\omega_m(\cdot,\tau) = \omega_m(\cdot,\tau); \qquad \bar v_m = \mathrm{K}*\bar\omega_m \end{cases} \tag{5.11}$$

Using $\bar v_m$ in (5.11), we can decompose v_m into the sum $v_m = \bar v_m + v_m^*$. In view of (5.11), $\bar v_m$ satisfies the following Navier-Stokes equation:

$$\partial_t \bar v_m + \bar v_m\cdot\nabla\bar v_m - \Delta\bar v_m = -\nabla\bar p_m, \tag{5.12}$$

where $\bar p_m$ is the pressure which satisfies the Poisson equation:

$$-\Delta\bar p_m = \partial_{ij}\big(\bar v_{m;i}\,\bar v_{m;j}\big). \tag{5.13}$$

Subtracting (5.12) from (4.27) yields

$$\partial_t v_m^* + v_m^*\cdot\nabla v_m^* - \Delta v_m^* = -v_m^*\cdot\nabla\bar v_m - \bar v_m\cdot\nabla v_m^* - \nabla p_m^* - \nabla\cdot\big(\nabla\phi_m\odot\nabla\phi_m\big), \quad \text{on } \mathbb{R}^2\times\big(\tau,\tau_{N'}\big). \tag{5.14}$$

By Calderon-Zygmund estimate and (3.27) in [3], it holds

$$\big\|\bar v_m(\cdot,t)\big\|_4 \lesssim \big\|\bar\omega_m(\cdot,t)\big\|_{4/3} \leqslant \big\|\omega_m(\cdot,\tau)\big\|_{\mathrm{L}^{4/3}\cap\,\mathrm{L}^1}, \qquad \forall\, t>\tau.$$

In light of (5.8), we get

$$\big\|\bar v_m(\cdot,t)\big\|_4 \leqslant c_1, \qquad \forall\, t>\tau. \tag{5.15}$$

Since $(\bar v_m, \bar p_m)$ satisfies (5.12)-(5.13) and (5.15), then by standard L^p-estimate for parabolic and elliptic equations (see [18]), any derivative of $\bar v_m$ is uniformly bounded on $\mathbb{R}^2\times[\tau+\epsilon,\infty)$ with the upper bound independent of m. Here $\epsilon>0$ is a constant arbitrarily given. Thus by Arzelà-Ascoli theorem, we can extract a subsequence, still denoted by $(\bar\omega_m,\bar v_m)$, so that $(\bar\omega_m,\bar v_m)$ and all its derivatives converge locally uniformly on $\mathbb{R}^2\times(\tau,\infty)$, as $m\to\infty$. The limit is denoted by $(\bar\omega_\infty,\bar v_\infty)$. Fixing the subsequence obtained and using the fact (see (3.27) in [3]) that

$$\big\|\bar\omega_m(\cdot,t)\big\|_q \leqslant \big\|\omega_m(\cdot,\tau)\big\|_{\mathrm{L}^q\cap\,\mathrm{L}^1}, \qquad \forall\, q\in[1,\infty] \text{ and } t>\tau, \tag{5.16}$$

we then have, for all $p>1$ and $t>\tau$, that the L^p-norm of $\bar\omega_m(\cdot,t)$ is uniformly bounded from above by a constant independent of m. Here we also used (5.8). Thus there is a subsequence, still denoted by $\bar\omega_m(\cdot,t)$, so that $\bar\omega_m(\cdot,t)$ converges weakly in L^p to a limit as $m\to\infty$. This limit must equal to $\bar\omega_\infty(\cdot,t)$ in the sense of distribution. It then follows that

$$\bar\omega_m(\cdot,t) \longrightarrow \bar\omega_\infty, \qquad \text{weakly in } \mathrm{L}^p\text{, for all } p>1 \text{ and } t>\tau. \tag{5.17}$$

In light of the local uniform convergence of $\bar v_m$ and (5.17), it holds, by the same derivations as for (4.31), that

$$\bar v_\infty = \mathrm{K}*\bar\omega_\infty. \tag{5.18}$$

Now we show that

$$(\bar\omega_\infty,\bar v_\infty) = (\bar\omega,\bar v), \tag{5.19}$$

where $\bar\omega$ is the unique mild solution of (1.3). Since $\bar\omega_m$ is the mild solution of (5.11), it can be represented by

$$\bar\omega_m(x,t) = \int_{\mathbb{R}^2}\mathrm{G}(x-z,t-\tau)\,\omega_m(z,\tau)\,\mathrm{d}z + \int_\tau^t\int_{\mathbb{R}^2}\nabla\mathrm{G}(x-z,t-s)\cdot\bar\omega_m(z,s)\bar v_m(z,s)\,\mathrm{d}z\,\mathrm{d}s. \tag{5.20}$$

Here (x,t) is an arbitrary point in $\mathbb{R}^2\times(\tau,\infty)$. Applying (3.28) in [3] yields

$$\|\bar{v}_m(\cdot,t)\|_\infty \lesssim_p \|\omega_m(\cdot,\tau)\|_{L^1\cap L^p}, \quad \forall\, p>2 \text{ and } t\geqslant\tau. \tag{5.21}$$

Therefore by (5.8), (5.16) and (5.21), $\bar{\omega}_m$ and $\bar{v}_m$ are uniformly bounded on $\mathbb{R}^2\times[\tau,\infty)$. Employing this uniform boundedness result and the fact that $(\bar{\omega}_m,\bar{v}_m,\omega_m(\cdot,\tau))$ converges to $(\bar{\omega}_\infty,\bar{v}_\infty,\omega(\cdot,\tau))$ pointwisely, by Lebesgue's dominated convergence theorem, we can take $m\to\infty$ in (5.20) and get

$$\bar{\omega}_\infty(x,t)=\int_{\mathbb{R}^2} G(x-z,t-\tau)\,\omega(z,\tau)\,dz+\int_\tau^t\int_{\mathbb{R}^2}\nabla G(x-z,t-s)\cdot\bar{\omega}_\infty(z,s)\,\bar{v}_\infty(z,s)\,dz\,ds. \tag{5.22}$$

Moreover it also follows that $(\bar{\omega}_\infty,\bar{v}_\infty)$ are uniformly bounded on $\mathbb{R}^2\times[\tau,\infty)$. Using the same derivation as for (5.16) and (5.21), we can also show that $(\bar{\omega},\bar{v})$ are uniformly bounded on $\mathbb{R}^2\times[\tau,\infty)$ since $\omega(\cdot,\tau)\in L^p$, for all $p\in[1,\infty]$ (see (5.10)). Using these two uniform boundedness results, (5.18) and (5.22), we can easily show that (5.19) holds, by a similar fashion as the proof of Lemma 4.2 in [3]. Here one just needs to know that $\bar{\omega}$ and $\bar{\omega}_\infty$ share same initial data at $t=\tau$. In light that for all $i=0,1,2,...$, we have $\nabla^i v_m\to\nabla^i v$ and $\nabla^i\bar{v}_m\to\nabla^i\bar{v}$ pointwisely, as $m\to\infty$, it then turns out that

$$\nabla^i v_m^*\longrightarrow\nabla^i v^*, \qquad \text{pointwisely on } \mathbb{R}^2\times[\tau,\tau_{N'}], \text{ for all } i=0,1.2,... \tag{5.23}$$

Here $v^*=v-\bar{v}$.

Step 6. This step is devoted to studying the uniform boundedness of the kinetic energy of v_m^* (see (5.31)). In order to use Lemma 5.1, we need ϕ_m, $\bar{v}_m$, v_m^* and p_m^* satisfy the assumption (5.2). Here p_m^* is the pressure in (5.14). By (5.9), $\nabla\phi_m$ satisfies the corresponding assumption in (5.2). In the following we consider $\bar{v}_m$, v_m^* and p_m^*.

(I). Estimate of $\bar{v}_m$. The L^∞-estimate of $\bar{v}_m$ is obtained in (5.21). Now we consider the estimate for $\nabla\bar{v}_m$. Using (3.29) in [3], one can find a $\tau_*>\tau$, where $\tau_*-\tau$ depends on the $L^1\cap L^p$-norm of $\omega_m(\cdot,\tau)$, such that

$$\|\nabla\bar{\omega}_m(\cdot,t)\|_{L^1\cap L^p}\lesssim_p (t-\tau)^{-1/2}\,\|\omega_m(\cdot,\tau)\|_{L^1\cap L^p}, \qquad \forall\, t\in(\tau,\tau_*).$$

In light of (5.8), $\tau_*-\tau$ can be independent of m. Therefore it follows that

$$\|\nabla\bar{v}_m(\cdot,t)\|_\infty\lesssim_p\|\nabla\bar{\omega}_m(\cdot,t)\|_{L^1\cap L^p}\lesssim_p(t-\tau)^{-1/2}\,\|\omega_m(\cdot,\tau)\|_{L^1\cap L^p}, \qquad \forall\, t\in(\tau,\tau_*).$$

By (3.38) in [3], we also have

$$\|\nabla\bar{v}_m(\cdot,t)\|_\infty\leqslant c\big(\|\omega_m(\cdot,\tau)\|_1\big)\,(t-\tau)^{-1}\leqslant c\big(\|\omega_m(\cdot,\tau)\|_1\big)\,(\tau_*-\tau)^{-1}, \qquad \forall\, t>\tau_*.$$

Thus the above two estimates imply that

$$\begin{aligned}\int_\tau^{\tau_{N'}}\|\nabla\bar{v}_m(\cdot,t)\|_\infty &= \int_\tau^{\tau_{N'}\wedge\tau_*}\|\nabla\bar{v}_m(\cdot,t)\|_\infty+\int_{\tau_{N'}\wedge\tau_*}^{\tau_{N'}}\|\nabla\bar{v}_m(\cdot,t)\|_\infty\\ &\lesssim_p \|\omega_m(\cdot,\tau)\|_{L^1\cap L^p}+c\big(\|\omega_m(\cdot,\tau)\|_1\big)\,(\tau_*-\tau)^{-1}\leqslant c_3,\end{aligned} \tag{5.24}$$

where c_3 is a positive constant independent of m.

(II). Estimate of v_m^*. Since $\omega_m(\cdot,\tau)\in C_2^{*,2}(\mathbb{R}^2)$ and $(\bar{\omega}_m,\bar{v}_m)$ satisfies (5.11), then by similar arguments as the proof of Theorem 1.2, we can show that $\bar{\omega}_m\in C_2^{*,2}[\tau,\tau+\delta]$, for some $\delta>0$ suitably small. In view of (5.8) and (5.21), similar arguments as in Step 3 of the proof for Theorem 1.3 can be applied to show that $\bar{\omega}_m\in C_2^{*,2}[\tau,\tau+\delta]$, for all $\delta>0$. Using the exponential decay of $\bar{\omega}_m$ at spatial infinity, by (5.11), we have

$$\int_{\mathbb{R}^2}\bar{\omega}_m(x,t)\,dx=\int_{\mathbb{R}^2}\omega_m(x,\tau)\,dx, \qquad \forall\, t\in[\tau,\tau_{N'}]. \tag{5.25}$$

As for ω_m, since it satisfies

$$\partial_t \omega_m + v_m \cdot \nabla \omega_m - \Delta \omega_m = -\nabla \times \nabla \cdot \big(\nabla \phi_m \odot \nabla \phi_m \big),$$

then we can integrate the above equation over $\mathbb{R}^2$ and get

$$\int_{\mathbb{R}^2} \omega_m(x,t)\,\mathrm{d}x = \int_{\mathbb{R}^2} \omega_m(x,\tau)\,\mathrm{d}x, \qquad \forall\, t \in [\tau, \tau_{N'}]. \tag{5.26}$$

Here we have used the exponential decay of $\nabla^i \phi_m$ for $i = 1, 2, 3$. (5.25)-(5.26) imply that

$$\int_{\mathbb{R}^2} \omega_m(x,t) - \bar{\omega}_m(x,t)\,\mathrm{d}x = 0, \qquad \forall\, t \in [\tau, \tau_{N'}].$$

Since for all $t \in [\tau, \tau_{N'}]$, $\omega_m(\cdot,t) - \bar{\omega}_m(\cdot,t)$ decays exponentially at spatial infinity. Moreover the $|||\cdot|||_2$-norm of $\omega_m(\cdot,t) - \bar{\omega}_m(\cdot,t)$ are uniformly bounded by the norm of $\omega_m - \bar{\omega}_m$ in $\mathrm{C}^*_2[\tau, \tau_{N'}]$, for all $t \in [\tau, \tau_{N'}]$. Therefore we can apply Lemma 5.2 to $v^*_m = \mathrm{K} * \omega_m - \mathrm{K} * \bar{\omega}_m$ and show that

$$\big\| v^*_m(\cdot,t) \big\|_2 \;\leqslant\; c_4 \Big(|||\, \omega_m - \bar{\omega}_m \,|||_{2;\,[\tau,\tau_{N'}]} \Big), \qquad \forall\, t \in [\tau, \tau_{N'}]. \tag{5.27}$$

The L^2-estimate of ∇v^*_m can be obtained by Calderon-Zygmund estimate as follows:

$$\int_{\mathbb{R}^2 \times \{t\}} \big| \nabla v^*_m \big|^2 \;\lesssim\; \int_{\mathbb{R}^2 \times \{t\}} \big| \nabla v_m \big|^2 + \int_{\mathbb{R}^2 \times \{t\}} \big| \nabla \bar{v}_m \big|^2 \;\lesssim\; \int_{\mathbb{R}^2 \times \{t\}} \big| \omega_m \big|^2 + \int_{\mathbb{R}^2 \times \{t\}} \big| \bar{\omega}_m \big|^2.$$

Here $t \in [\tau, \tau_{N'}]$ is arbitrarily given. Applying Lemma 2.4 and (5.16) to the most-right-hand side above, we get

$$\int_{\mathbb{R}^2 \times \{t\}} \big| \nabla v^*_m \big|^2 \;\leqslant\; c_5 + |||\omega_m|||^2_{2;\,[\tau,\tau_{N'}]}, \qquad \forall\, t \in (\tau, \tau_{N'}). \tag{5.28}$$

Here c_5 is a positive constant depending on τ and the initial data (ϕ_0, ω_0).

(III). Estimate of p^*_m. Since v^*_m is divergent free, it then follows by (5.14) that

$$-\Delta p^*_m = \partial_{ij} \big(v^*_{m;i}\, v^*_{m;j} \big) + 2\, \partial_{ij} \big(\bar{v}_{m;i}\, v^*_{m;j} \big) + \mathrm{div} \big(\nabla \cdot \big(\nabla \phi_m \odot \nabla \phi_m \big) \big).$$

Therefore Calderon-Zygmund estimate implies that

$$\big\| p^*_m \big\|_{4/3} \;\lesssim\; \big\| v^*_m \big\|^2_{8/3} + \big\| \, |\bar{v}_m| \, |v^*_m| \, \big\|_{4/3} + \big\| \nabla \phi_m \big\|^2_{8/3}. \tag{5.29}$$

On the other hand p^*_m can be represented by

$$p^*_m(x,t) = -(2\pi)^{-1} \int_{\mathbb{R}^2} \frac{x_j - z_j}{|x-z|^2} \Big[v^*_m \cdot \nabla v^*_{m,j} + 2\, \bar{v}_m \cdot \nabla v^*_{m,j} + \partial_j \phi_m \cdot \Delta \phi_m \Big]_{(z,t)} \mathrm{d}z.$$

Still by Calderon-Zygmund estimate, it follows that

$$\big\| \nabla p^*_m \big\|_{4/3} \;\lesssim\; \big\| v^*_m \cdot \nabla v^*_m \big\|_{4/3} + \big\| \bar{v}_m \cdot \nabla v^*_m \big\|_{4/3} + \big\| \nabla \phi_m \cdot \Delta \phi_m \big\|_{4/3}. \tag{5.30}$$

In light of (5.9), (5.15), (5.27)-(5.28), one then can apply Ladyzhenskaya's inequality and Hölder's inequality to the right-hand sides of (5.29)-(5.30) and show that $p^*_m \in \mathrm{L}^{4/3}\big([\tau, \tau_{N'}]; W^{1,4/3}\big)$.

Notice that (ϕ_m, v^*_m) satisfies (4.26) and (5.14) on $\mathbb{R}^2 \times (\tau, \tau_{N'})$. (5.9) and the above arguments in (I), (II), (III) show that ϕ_m, $\bar{v}_m$, v^*_m and p^*_m satisfy the assumption (5.2). Thus we can apply Lemma 5.1 to get, for all $t \in (\tau, \tau_{N'})$, that

$$\int_{\mathbb{R}^2 \times \{t\}} \big| v^*_m \big|^2 + \big| \nabla \phi_m \big|^2 + \int_\tau^t \int_{\mathbb{R}^2} \big| \nabla v^*_m \big|^2 + \big| \Delta \phi_m + |\nabla \phi_m|^2 \phi_m \big|^2 \;\leqslant\; \exp\Big\{ \mathrm{c} \int_\tau^t \big\| \nabla \bar{v}_m \big\|_\infty \Big\} \int_{\mathbb{R}^2 \times \{\tau\}} \big| \nabla \phi_m \big|^2.$$

Here we have used the fact that $v_m^*(\cdot,\tau) \equiv 0$. By (5.9) and (5.24), the right-hand side above is uniformly bounded by a constant independent of m. Therefore it follows that

$$\max_{t\in[\tau,\tau_{N'}]}\int_{\mathbb{R}^2\times\{t\}}\big|v_m^*\big|^2+\big|\nabla\phi_m\big|^2+\int_\tau^{\tau_{N'}}\int_{\mathbb{R}^2}\big|\nabla v_m^*\big|^2+\big|\Delta\phi_m+|\nabla\phi_m|^2\phi_m\big|^2 \leqslant c_6, \quad \forall\, m>N'. \tag{5.31}$$

Here $c_6>0$ is a constant independent of m.

Step 7. In view of (5.23) and the pointwise convergence of $\nabla^i\phi_m$ $(i=0,1,2)$, by Fatou's lemma, we can take $m\to\infty$ in (5.31) and get

$$\max_{t\in[\tau,\tau_{N'}]}\int_{\mathbb{R}^2\times\{t\}}\big|v^*\big|^2+\big|\nabla\phi\big|^2+\int_\tau^{\tau_{N'}}\int_{\mathbb{R}^2}\big|\nabla v^*\big|^2+\big|\Delta\phi+|\nabla\phi|^2\phi\big|^2 \leqslant c_6,$$

which furthermore implies

$$v^*\in \mathrm{L}^\infty\big([\tau,\tau_{N'}];\mathrm{L}^2\big)\cap\mathrm{L}^2\big([\tau,\tau_{N'}];\mathrm{H}^1\big). \tag{5.32}$$

As $m\to\infty$, $\bar v_m\to\bar v$ and $\nabla\bar v_m\to\nabla\bar v$ pointwisely. Then by (5.15), (5.21), (5.24) and Fatou's lemma, we have

$$\max_{t\in[\tau,\tau_{N'}]}\big\|\bar v(\cdot,t)\big\|_4+\max_{t\in[\tau,\tau_{N'}]}\big\|\bar v(\cdot,t)\big\|_\infty+\int_\tau^{\tau_{N'}}\big\|\nabla\bar v\big\|_\infty < \infty. \tag{5.33}$$

In light of (5.10), (5.32)-(5.33), by the same arguments as in (III) of Step 6, we know that

$$p^*\in\mathrm{L}^{4/3}\big([\tau,\tau_{N'}];W^{1,4/3}\big). \tag{5.34}$$

Here p^* is the pressure in the following equation:

$$\partial_t v^*+v^*\cdot\nabla v^*-\Delta v^*=-v^*\cdot\nabla\bar v-\bar v\cdot\nabla v^*-\nabla p^*-\nabla\cdot\big(\nabla\phi\odot\nabla\phi\big), \quad \text{on } \mathbb{R}^2\times\big(\tau,\tau_{N'}\big). \tag{5.35}$$

The derivation of this equation is the same as (5.14). One just needs to know that (ϕ,v) satisfies the second equation in (1.1) and $(\bar\omega,\bar v)$ solves (1.3) in Theorem 1.3. (5.10), (5.32)-(5.34) imply that ϕ, $\bar v$, v^* and p^* satisfy the assumption (5.2). Recalling that (ϕ,v^*) satisfies the first equation in (1.2) and (5.35) above, we then obtain the global energy inequality (1.5) in Theorem 1.3, with an use of Lemma 5.1. Here we take $T_*=\tau_{N'}$ in Theorem 1.3. Noticing that $v^*(\cdot,\tau)\equiv 0$, then we have, by taking $t_1=\tau$ in (1.5), that

$$\int_{\mathbb{R}^2\times\{t\}}\big|v^*\big|^2+\big|\nabla\phi\big|^2+\int_\tau^t\int_{\mathbb{R}^2}\big|\nabla v^*\big|^2+\big|\Delta\phi+|\nabla\phi|^2\phi\big|^2 \leqslant \exp\left\{\mathrm{c}\int_\tau^t\big\|\nabla\bar v\big\|_\infty\right\}\int_{\mathbb{R}^2\times\{\tau\}}\big|\nabla\phi\big|^2,$$

for all t satisfying $\tau<t<\tau_{N'}$. Since at $t=\tau$, $\omega(\cdot,\tau)\in\mathrm{L}^1\cap\mathrm{L}^p$ for all $p>2$, then we know, by similar arguments as for (5.24), that $\big\|\nabla\bar v\big\|_\infty$ is L^1-integrable on $[\tau,\tau_{N'}]$. Therefore we can take $t\to\tau^+$ in the above estimate and get

$$\limsup_{t\to\tau^+}\int_{\mathbb{R}^2\times\{t\}}\big|v^*\big|^2+\big|\nabla\phi\big|^2 \leqslant \int_{\mathbb{R}^2\times\{\tau\}}\big|\nabla\phi\big|^2.$$

Since $\nabla\phi(\cdot,t)$ converges weakly in L^2 to $\nabla\phi(\cdot,\tau)$, as $t\to\tau^+$, by lower semi-continuity of the L^2-norm, we further have from the last inequality that

$$\int_{\mathbb{R}^2\times\{\tau\}}\big|\nabla\phi\big|^2 \leqslant \limsup_{t\to\tau^+}\int_{\mathbb{R}^2\times\{t\}}\big|v^*\big|^2+\big|\nabla\phi\big|^2 \leqslant \int_{\mathbb{R}^2\times\{\tau\}}\big|\nabla\phi\big|^2.$$

This shows that $v^*(\cdot,t)\to 0$ and $\nabla\phi(\cdot,t)\to\nabla\phi(\cdot,\tau)$ strongly in L^2 as $t\to\tau^+$. Same strong convergence and also the decomposition (1.4) hold when $\tau=0$, provided that we know $\omega_0\in\mathrm{L}^1\cap\mathrm{L}^p$ for some $p>1$. Indeed we just need to check the L^1-integrability of $\big\|\nabla\bar v\big\|_\infty$ near $t=0$. Without loss of generality, in the following arguments, we assume $p \in (1,2)$. Moreover we let $(\bar\omega, \bar v)$ satisfy (1.3) with $\tau = 0$ there. For any $(x,t) \in \mathbb{R}^2 \times (0,\infty)$, it holds

$$\nabla \bar\omega(x,t) = \nabla \mathrm{G} * \omega_0 - \int_0^t \int_{\mathbb{R}^2} \nabla \mathrm{G}(x-z, t-s) \big(\bar v(z,s) \cdot \nabla_z \big) \bar\omega(z,s) \, \mathrm{d}z \, \mathrm{d}s.$$

By Minkowski's and Young's inequality, it follows, for all

$$q \in \left(\max\left\{ p, \, \frac{2p}{3p-2} \right\}, \, \frac{2p}{2-p} \right), \tag{5.36}$$

that

$$\begin{aligned} \| \nabla \bar\omega(\cdot,t) \|_q \quad &\leqslant \quad \| \nabla \mathrm{G} * \omega_0 \|_q + \int_0^t \| \nabla \mathrm{G}(\cdot, t-s) \|_{\frac{2pq}{3pq-2q}} \| \bar v \cdot \nabla \bar\omega \|_{\frac{2pq}{2p+2q-pq}}(s) \, \mathrm{d}s \\ &\lesssim_{p,q} \quad \| \nabla \mathrm{G}(\cdot,t) \|_{\frac{pq}{pq+p-q}} \| \omega_0 \|_p + \int_0^t (t-s)^{-1/p} \| \bar v \|_{\frac{2p}{2-p}} \| \nabla \bar\omega \|_q. \end{aligned}$$

Applying Calderon-Zygmund estimate and (3.27) in [3], we get from the above estimate that

$$\begin{aligned} \| \nabla \bar\omega(\cdot,t) \|_q \quad &\lesssim_{p,q} \quad t^{-1/2+1/q-1/p} \| \omega_0 \|_p + \int_0^t (t-s)^{-1/p} \| \bar\omega \|_p \| \nabla \bar\omega \|_q \\ &\lesssim \quad t^{-1/2+1/q-1/p} \| \omega_0 \|_p + \| \omega_0 \|_{\mathrm{L}^p \cap \mathrm{L}^1} \int_0^t (t-s)^{-1/p} \| \nabla \bar\omega \|_q. \end{aligned} \tag{5.37}$$

Now we denote by $A_q^*(\cdot)$ the quantity:

$$A_q^*(t) = \max_{0<\tau<t} \tau^{1/2-1/q+1/p} \| \nabla \bar\omega(\cdot,\tau) \|_q.$$

Then (5.37) can be reduced to

$$A_q^*(t) \quad \lesssim_{p,q} \quad \| \omega_0 \|_p + A_q^*(t) \, t^{1-1/p} \| \omega_0 \|_{\mathrm{L}^p \cap \mathrm{L}^1}.$$

Therefore we can find a T small enough (smallness depends on p, q and $\| \omega_0 \|_p$) such that

$$\| \nabla \bar\omega(\cdot,t) \|_q \quad \lesssim_{p,q} \quad t^{-1/2+1/q-1/p} \| \omega_0 \|_p, \qquad \forall \, t \in \big(0, T\big]. \tag{5.38}$$

In light of

$$\begin{aligned} \nabla \bar v(x,t) \quad &= \quad \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-z)^\perp}{|x-z|^2} \nabla \bar\omega(z,t) \, \mathrm{d}z \\ &= \quad \frac{1}{2\pi} \int_{|x-z|<1} \frac{(x-z)^\perp}{|x-z|^2} \nabla \bar\omega(z,t) \, \mathrm{d}z + \int_{|x-z|\geqslant 1} \frac{(x-z)^\perp}{|x-z|^2} \nabla \bar\omega(z,t) \, \mathrm{d}z, \end{aligned}$$

then for $q_1 \in (1,2)$ and $q_2 > 2$ satisfying (5.36), it holds

$$\begin{aligned} \big| \nabla \bar v(x,t) \big| \quad &\lesssim \quad \int_{|x-z|<1} \frac{1}{|x-z|} \big| \nabla \bar\omega \big|(z,t) \, \mathrm{d}z + \int_{|x-z|\geqslant 1} \frac{1}{|x-z|} \big| \nabla \bar\omega \big|(z,t) \, \mathrm{d}z \\ &\leqslant \quad \| \nabla \bar\omega(\cdot,t) \|_{q_2} \left(\int_{|x-z|<1} \frac{1}{|x-z|^{q_2'}} \, \mathrm{d}z \right)^{1/q_2'} + \| \nabla \bar\omega(\cdot,t) \|_{q_1} \left(\int_{|x-z|\geqslant 1} \frac{1}{|x-z|^{q_1'}} \, \mathrm{d}z \right)^{1/q_1'} \\ &\lesssim_{q_1,q_2} \quad \| \nabla \bar\omega(\cdot,t) \|_{q_2} + \| \nabla \bar\omega(\cdot,t) \|_{q_1}. \end{aligned}$$

Here q_1' and q_2' are Hölder conjugates of q_1 and q_2, respectively. Applying (5.38) to the last estimate yields

$$\| \nabla \bar v(\cdot,t) \|_\infty \;\; \lesssim_{p,\,q_1,\,q_2} \;\; \| \omega_0 \|_p \, t^{-1/2+1/q_2-1/p} + \| \omega_0 \|_p \, t^{-1/2+1/q_1-1/p}, \qquad \forall \, t \in \big(0, T\big].$$

Therefore by (5.36), $\| \nabla \bar v(\cdot,t) \|_\infty$ is integrable near $t = 0$. The proof is finished. □

In the end we finish this article by a proof of Theorem 1.4.

Proof of Theorem 1.4. In view of Theorem 1.3, (1.1) can be solved on $\big(0, T_*\big)$. Moreover the velocity v satisfies the decomposition (1.4). Since (ϕ, v^*) solves the system:

$$\begin{cases} \partial_t \phi + v^* \cdot \nabla \phi - \Delta \phi = -\bar{v} \cdot \nabla \phi + |\nabla \phi|^2 \phi \,, & \text{on } \mathbb{R}^2 \times (\tau, T_*); \\ \partial_t v^* + v^* \cdot \nabla v^* - \Delta v^* = -v^* \cdot \nabla \bar{v} - \bar{v} \cdot \nabla v^* - \nabla p^* - \nabla \cdot \big(\nabla \phi \odot \nabla \phi\big), & \text{on } \mathbb{R}^2 \times \big(\tau, T_*\big); \\ \operatorname{div} \bar{v} = \operatorname{div} v^* = 0. \end{cases} \tag{5.39}$$

and $\bar{v}$ already exists on the whole space $\mathbb{R}^2 \times (\tau, \infty)$, we only need to extend (ϕ, v^*) globally in time so that the extended (ϕ, v^*) solves (5.39) weakly on $\mathbb{R}^2 \times (\tau, \infty)$. In light of the global energy estimate (1.5), we can use similar arguments as the proofs of Theorems 1.2-1.3 and Lemma 5.2 in [20] to obtain such extension of (ϕ, v^*). Details of the proof are omitted for brevity. □

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