THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 3 (September 30)

1 More on Limit of Sequences

Example 1. Let (y_n) be a sequence of positive numbers such that $\lim_n y_n = 2$. By virtue of ε-N terminology, show that

$$\lim_n \frac{y_n}{y_n^2 - 3} = 2.$$

Solution. Let $\varepsilon > 0$ be given. For $n \in \mathbb{N}$,

$$\left|\frac{y_n}{y_n^2-3} - 2\right| = \left|\frac{y_n - 2y_n^2 + 6}{y_n^2 - 3}\right| = \left|\frac{(2y_n+3)(y_n-2)}{y_n^2-3}\right|$$
$$= \frac{|2y_n+3|}{|y_n^2-3|} \cdot |y_n - 2|.$$

Want: a positive lower bound of $|y_n^2 - 3|$ when n is large.

To archive this, we choose a neighbourhood of 2 that avoids the zeros of $y^2 - 3$, that is $\pm\sqrt{3} \approx \pm 1.73$. For example, $(2 - \frac{1}{4}, 2 + \frac{1}{4})$.

If $|y_n - 2| < \dfrac{1}{4}$, then

$$\frac{7}{4} < y_n < \frac{9}{4} \implies \frac{1}{16} < y_n^2 - 3 < \frac{33}{16},$$

and

$$|2y_n + 3| = |2(y_n - 2) + 7| \le 2|y_n - 2| + 7 \le 2(1) + 7 = 9.$$

Combining the two bounds, we have

$$|y_n - 2| < \frac{1}{4} \implies \left|\frac{y_n}{y_n^2-3} - 2\right| \le \frac{9}{\frac{1}{16}}|y_n - 2| = 144|y_n - 2|.$$

Take $\varepsilon' := \min\left\{\dfrac{1}{4}, \dfrac{\varepsilon}{144}\right\}$. Since $\lim_n y_n = 2$, there exists $N \in \mathbb{N}$ such that

$$|y_n - 2| < \varepsilon' \qquad \text{for all } n \ge N.$$

Now, for $n \ge N$, we have

$$\left|\frac{y_n}{y_n^2-3} - 2\right| \le 144|y_n - 2| < 144\varepsilon' \le \varepsilon.$$

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Example 2. Let (x_n) be a sequence of real numbers. Define

$$s_n = \frac{x_1 + x_2 \cdots + x_n}{n} \qquad \text{for all } n \in \mathbb{N}.$$

(a) If $\lim(x_n) = \ell$, where $\ell \in \mathbb{R}$, show that $\lim(s_n) = \ell$.

(b) Is the converse of (a) true?

Solution. (a) Without loss of generality, we assume that $\ell = 0$. (This can be done by letting $y_n = x_n - \ell$.)

For $1 \le m < n$, we separate the s_n into two parts:

$$s_n = \frac{x_1 + \cdots + x_m}{n} + \frac{x_{m+1} + \cdots + x_n}{n}.$$

In order to show that $|s_n|$ is small when n is large, we will use different approaches to estimate the size of the first and second part.

Since (x_n) is convergent, it is bounded, so we can find $M > 0$ such that

$$|x_n| \le M \qquad \text{for all } n \in \mathbb{N}.$$

Let $\varepsilon > 0$ be given. Since $\lim(x_n) = 0$, there exists $m \in \mathbb{N}$ such that

$$|x_n| < \varepsilon/2 \qquad \text{for all } n \ge m.$$

By Archimedean Property, choose $N \in \mathbb{N}$ such that $N > \max\left\{\frac{mM}{\varepsilon/2}, m\right\}$.

Now, for $n \ge N$, we have

$$\begin{aligned} |s_n| &\le \frac{|x_1| + \cdots + |x_m|}{n} + \frac{|x_{m+1}| + \cdots + |x_n|}{n} \\ &< \frac{mM}{n} + \frac{(n-m)\varepsilon/2}{n} \\ &\le \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

Hence $\lim(s_n) = 0$.

(b) No. Consider $x_n := (-1)^n$. Then $s_n = \begin{cases} -\frac{1}{n} & n \text{ odd,} \\ 0 & n \text{ even.} \end{cases}$

Hence $\lim(s_n) = 0$ while (x_n) diverges.

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2 Monotone Sequences

Monotone Convergence Theorem. *A monotone sequence of real numbers is convergent if and only if it is bounded. Furthermore,*

(a) If (x_n) is a bounded increasing sequence, then $\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$.

(b) If (y_n) is a bounded decreasing sequence, then $\lim(y_n) = \inf\{y_n : n \in \mathbb{N}\}$.

Example 3. Let (x_n) be the sequence defined by

$$x_1 := 10, \qquad x_{n+1} := \frac{2x_n}{x_n^2 + 1} \qquad \text{for } n \geq 1.$$

Show that (x_n) is convergent and find its limit.

Solution. Note that, for $n \geq 1$,

$$1 - x_{n+1} = 1 - \frac{2x_n}{x_n^2+1} = \frac{1 - 2x_n + x_n^2}{x_n^2+1} = \frac{(x_n-1)^2}{x_n^2+1} \geq 0.$$

Furthermore, it follows easily from induction that $x_n \geq 0$ for all $n \in \mathbb{N}$. Therefore $0 \leq x_n \leq 1$ for $n \geq 2$.

Next we prove that $x_{n+1} \geq x_n$ for $n \geq 2$. First observe that

$$x_2 = \frac{20}{101} < \frac{2(\frac{20}{101})}{(\frac{20}{101})^2 + 1} = x_3.$$

If we assume that the inequality is true for $n = k$, where $k \geq 2$, then

$$\begin{aligned} x_{k+2} - x_{k+1} &= \frac{2x_{k+1}}{x_{k+1}^2+1} - \frac{2x_k}{x_k^2+1} \\ &= \frac{2x_{k+1}x_k^2 + 2x_{k+1} - 2x_k x_{k+1}^2 - 2x_k}{(x_{k+1}^2+1)(x_k^2+1)} \\ &= \frac{2(x_{k+1}-x_k)(1 - x_k x_{k+1})}{(x_{k+1}^2+1)(x_k^2+1)} \geq 0. \end{aligned}$$

so that the inequality is also true for $n = k+1$.

Therefore, it follows from Mathematical Induction that the $x_{n+1} \geq x_n$ for $n \geq 2$.

As (x_n) is bounded increasing, Monotone Convergence Theorem implies that (x_n) is convergent. Let $\lim(x_n) = \ell$. Then

$$\ell = \frac{2\ell}{\ell^2+1} \implies \ell(\ell-1)(\ell+1) = 0.$$

Thus $\ell = 1$ since $\ell = 0$ and $\ell = -1$ are rejected. Hence $\lim(x_n) = 1$.

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