THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 2 (September 23, 25)

1 Applications of the Supremum Property

Definition. Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Then $u \in \mathbb{R}$ is said to be a **supremum** of S if

(i) $s \leq u$ for all $s \in S$;

(ii) for any $\varepsilon > 0$, there exists $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.

The Completeness Property of $\mathbb{R}$. *Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.*

Archimedean Property. *If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x \leq n_x$.*

Example 1 (Existence of $\sqrt[n]{a}$)**.** Let $a > 0$. Show that for any $n \in \mathbb{N}$, there exists a unique positive number x such that $x^n = a$.

Solution. (Uniqueness) Clear because if $0 < a < b$, then $a^n < b^n$.

(Existence) Let $S := \{s \in \mathbb{R} : s \geq 0, s^n < a\}$. Note that

(i) $S \neq \emptyset$ since $0 \in S$;

(ii) S is bounded above since $s > (1 + a) \implies s^n > (1 + a)^n > na > a$.

By the completeness property, S has a supremum. Let $x := \sup S$. Clearly $x \geq 0$. If we can show that $x^n = a$, then we must have $x > 0$. To prove $x^n = a$, we eliminate the cases $x^n < a$ and $x^n > a$.

We will make use of the following elementary inequality: if $0 \leq a \leq b$, then

$$b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1}) \leq (b - a)nb^{n-1}.$$

Case 1: Suppose $x^n < a$

Want: $\left(x + \frac{1}{m}\right)^n < a$ for some large m.

Note that

$$\left(x + \frac{1}{m}\right)^n - x^n \leq \frac{1}{m} n \left(x + \frac{1}{m}\right)^{n-1} \leq \frac{1}{m} n (x + 1)^{n-1}.$$

By A.P. there exists $m \in \mathbb{N}$ such that $\dfrac{1}{m} < \dfrac{a - x^n}{n(x + 1)^{n-1}}$.

Now $0 \leq x < x + \dfrac{1}{m}$ and $\left(x + \dfrac{1}{m}\right)^n < a$, contradicting the fact that x is an upper bound of S.

Case 2: Suppose $x^n > a$

Want: $\left(x - \frac{1}{m}\right)^n > a$ for some large m.

By A.P. there exists $m \in \mathbb{N}$ such that $\dfrac{1}{m} < \dfrac{x^n - a}{nx^{n-1}} < x$. Then $x - \dfrac{1}{m} > 0$, and hence

$$x^n - \left(x - \frac{1}{m}\right)^n \leq \frac{1}{m} n x^{n-1} < x^n - a.$$

Now $t > x - \dfrac{1}{m} \implies t^n > \left(x - \frac{1}{m}\right)^n > a \implies t \notin S$, i.e. $t \leq x - \dfrac{1}{m}$ for all $t \in S$, contradicting the fact that x is the least upper bound of S.

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2 Limit of Sequences

Definition. A sequence $X = (x_n)$ in $\mathbb{R}$ is said to be converge to $x \in \mathbb{R}$, or x is said to be a limit of (x_n), if for every $\varepsilon > 0$ there exists a natural number $K(\varepsilon)$ such that for all $n \geq K(\varepsilon)$, the terms x_n satisfy $|x_n - x| < \varepsilon$.

Procedure. *To show that* $\lim(x_n) = x$*, we proceed as follow:*

1. *Let* $\varepsilon > 0$ *be given. (*ε *is arbitrary, but cannot be changed once fixed.)*
2. *Find a useful estimate for* $|x_n - x|$*.*
3. *Find* $K(\varepsilon) \in \mathbb{N}$ *such that the estimate in 2 is less than* ε *whenever* $n \geq K(\varepsilon)$*.*
4. *Complete the proof.*

Example 2. Use the definition of the limit of a sequence to show $\lim \left(\dfrac{n^2 - n}{2n^2 + 3}\right) = \dfrac{1}{2}$.

Solution.

1. Fix an arbitrary $\varepsilon > 0$. It cannot be changed once fixed.

Let $\varepsilon > 0$ be given.

2. Find a useful estimate for $|x_n - x|$.

For $n \geq 1$,

$$\begin{aligned} \left| \frac{n^2 - n}{2n^2 + 3} - \frac{1}{2} \right| &= \left| \frac{2n^2 - 2n - 2n^2 - 3}{2(2n^2 + 3)} \right| = \frac{2n + 3}{2(2n^2 + 3)} \\ &\leq \frac{2n + 3}{n^2} \\ &\leq \frac{2n + 3n}{n^2} = \frac{5}{n}. \end{aligned}$$

Do not try to solve $\dfrac{2n+3}{2(2n^2+3)} < \varepsilon$ directly.

3. Find $K = K(\varepsilon) \in \mathbb{N}$ such that the estimate above is less than ε whenever $n \geq K$.

Let $K := \lfloor 5/\varepsilon \rfloor + 1$.

4. Complete the argument.

Now, for all $n \geq K$, we have

$$\left| \frac{n^2 - n}{2n^2 + 3} - \frac{1}{2} \right| \leq \frac{5}{n} \leq \frac{5}{K} < \varepsilon.$$

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Example 3. Use the definition of limit to show that $\lim(\sqrt{n+1} - \sqrt{n}) = 0$.

Example 4. Let (x_n) be a sequence given by $x_n := 1+(-1)^n$. Show that (x_n) is divergent.

Solution. Suppose on the contrary that (x_n) converges. Assume $\lim(x_n) = \ell \in \mathbb{R}$. Then for $\varepsilon_0 = 1/2$, there exists $K \in \mathbb{N}$ such that $|x_n - \ell| < \varepsilon_0$ whenever $n \geq K$. In particular,

$$|x_K - x_{K+1}| = |(x_K - \ell) - (x_{K+1} - \ell)| \leq |x_K - \ell| + |x_{K+1} - \ell| < \varepsilon_0 + \varepsilon_0 = 1. \qquad (*)$$

However,

$$|x_K - x_{K+1}| = |(1 + (-1)^K) - (1 + (-1)^{K+1})| = |(-1)^K - (-1)^{K+1}| = 2,$$

contradicting $(*)$. Thus (x_n) is divergent. ◀