THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 1 (September 16, 18)

1 Negation and Quantifiers

Example 1. Negate the following statements.

(a) n is a prime number between 1 and 10.

(b) If n^2 is divisible by 4, then n is divisible by 2.

(c) For any real number x, $x^2 \geq 0$.

(d) For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $1/N < \varepsilon$.

Solution. (a) n is not a prime number or $n < 1$ or $n > 10$.

(b) n^2 is divisible by 4 but n is not divisible by 2.

(c) There exists a real number x such that $x^2 < 0$.

(d) There exists $\varepsilon > 0$ such that for any $N \in \mathbb{N}$, we have $1/N \geq \varepsilon$.

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2 Algebraic Properties of $\mathbb{R}$

The Field Axioms of $\mathbb{R}$. $(\mathbb{R}, +, \cdot)$ *satisfies the following properties:*

(A1) "$+$" *commutative*	**(M1)** "$\cdot$" *commutative*	**(D)** *distributive law*
(A2) "$+$" *associative*	**(M2)** "$\cdot$" *associative*	**(N)** $0 \neq 1$
(A3) "0"	**(M3)** "1", $1 \neq 0$	
(A4) "$+$" *inverse*	**(M1)** "$\cdot$" *inverse*	

Proposition 1. *(a)* $a \cdot 0 = 0 \cdot a = 0$ *for any* $a \in \mathbb{R}$.

(b) If $a + b = 0$*, then* $b = -a$.

Example 2. Let $a \in \mathbb{R}$. Show that $(-1)a = -a$.

Solution. From Proposition 1(b), we need to show that $(-1)a + a = 0$. Indeed,

$$\begin{aligned}
(-1)a + a &= (-1)a + 1 \cdot a && \text{(by M3)} \\
&= [(-1) + 1] \cdot a && \text{(by D)} \\
&= 0 \cdot a && \text{(by A4)} \\
&= 0 && \text{(by Proposition 1(a))}
\end{aligned}$$

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3 Order Properties of $\mathbb{R}$

The Order Properties of $\mathbb{R}$. *There is a nonempty subset $\mathbb{P}$ of $\mathbb{R}$, called the set of positive real numbers, that satisfies the following properties:*

(I) $a, b \in \mathbb{P} \implies a + b \in \mathbb{P}$,

(II) $a, b \in \mathbb{P} \implies ab \in \mathbb{P}$,

(III) If $a \in \mathbb{R}$, *then exactly one of the following holds:*

$$a \in \mathbb{P}, \qquad a = 0, \qquad -a \in \mathbb{P}.$$

Write $a > 0$ *if* $a \in \mathbb{P}$; *and write* $a > b$ *if* $a - b \in \mathbb{P}$.

Example 3. Let $a, b \in \mathbb{R}$. Show that if $0 < a < b$, then $1/b < 1/a$.

Solution. Note that $(ab)(1/b - 1/a) = a - b$.
If $1/a - 1/b = 0$, then $a - b = (ab) \cdot 0 = 0$, contradicting the assumption that $a < b$.
If $1/a - 1/b < 0$, then $1/b - 1/a > 0$, so that, by (II), $a - b = (ab)(1/b - 1/a) > 0$, which is again a contradiction.
Hence, by (III), $1/a - 1/b > 0$. That is $1/a > 1/b$. ◀

4 The Completeness Property of $\mathbb{R}$

Definition. Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Then $u \in \mathbb{R}$ is said to be a **supremum** of S if it satisfies the conditions

(i) u is an upper bound of S (that is, $s \leq u$ for all $s \in S$), and

(ii) if v is any upper bound of S, then $u \leq v$.

Here (ii) is equivalent to

(ii)' if $v < u$, then there exists $s_v \in S$ such that $v < s_v$.

Remarks. The **infimum** of a set S can be defined similarly. The supremum and infimum are unique and will be dented by $\sup S$ and $\inf S$, respectively.

Example 4. Find the infimum and supremum, if they exist, of the set $A := \{x \in \mathbb{R} : 1/x < x\}$. Justify your answers.

Solution. Note that

$$\frac{1}{x} < x \iff \frac{x^2 - 1}{x} = \frac{(x-1)(x+1)}{x} > 0 \iff x \in (-1, 0) \cup (1, \infty).$$

Thus $A = (-1, 0) \cup (1, \infty)$.

It is easy to see that A is not bounded above. For otherwise, if u is an upper bound of A, then

$$1 < |u| + 2 \implies |u| + 2 \in A \quad \text{and} \quad u \le |u| < |u| + 2.$$

Contradiction arises. So $\sup A$ does not exist.

Next we want to show that $\inf A = -1$. Clearly

$$x > -1 \quad \text{for all } x \in A.$$

So -1 is a lower bound of A.

Want: if $v > -1$, then v is not a lower bound of A, i.e. $\exists\, s_v \in A$ s.t. $s_v < v$.

Take $s_v := \min\{(v-1)/2, -1/2\}$. Then

$$-1 < s_v \le -1/2 < 0,$$

so that $s_v \in A$. Moreover,

$$s_v \le (v-1)/2 < (v+v)/2 = v.$$

Hence $\inf A = -1$. ◀

The Completeness Property of $\mathbb{R}$. *Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.*

Example 5. Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A + B := \{a + b : a \in A, b \in B\}$. Prove that

$$\sup(A + B) = \sup A + \sup B.$$

Solution. For $a \in A$, $b \in B$, we have $a \le \sup A$, $b \le \sup B$, so that

$$a + b \le \sup A + \sup B.$$

Hence $A + B$ is bounded above by $\sup A + \sup B$. By the completeness axiom, $\sup(A + B)$ exists and

$$\sup(A + B) \le \sup A + \sup B.$$

On the other hand, fix $b \in B$. Then, for $a \in A$,

$$a + b \le \sup(A + B) \implies a \le \sup(A + B) - b.$$

Hence RHS is an upper bounded of A, and thus

$$\sup A \le \sup(A + B) - b \implies b \le \sup(A + B) - \sup A. \tag{1}$$

Since (1) is true for any $b \in B$, RHS is an upper bound of B, and thus

$$\sup B \le \sup(A + B) - \sup A,$$

that is

$$\sup(A + B) \ge \sup A + \sup B.$$

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