lecture 4.

- Thm of continuity.

$$\lim_{z \to z_0} f \pm g = \lim_{z \to z_0} f \pm \lim_{z \to z_0} g$$

$$\lim_{z \to z_0} f g = \lim_{z \to z_0} f \cdot \lim_{z \to z_0} g$$

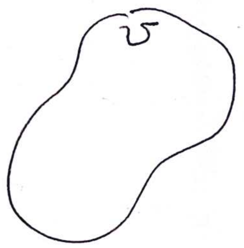
$$\lim_{z \to z_0} \frac{f}{g} = \lim_{z \to z_0} f \Big/ \lim_{z \to z_0} g. \qquad \lim_{z \to z_0} f(g(z)) = f\left(\lim_{z \to z_0} g(z)\right)$$

- Thm: $f = u + iv$ is Cont. $\Leftrightarrow$ u and v are both continuous.

ex: $x(t)$, $y(t)$ are 2 cont. single variable functions.

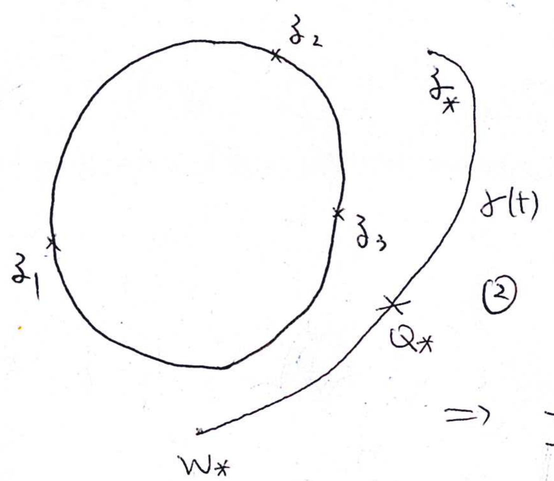
the $z(t) = x(t) + iy(t)$ define a Cont. path.

ex.



Ω is an open set. f is cont in Ω. $z(t)$ is a cont path in Ω. then $f(z(t))$ is a cont. function

ex.



① For all z_* outside the Circle.

$$\frac{z - z_1}{z - z_2} \Big/ \frac{z_3 - z_1}{z_3 - z_2} \text{ is cont at } z_*.$$

② $\mathrm{Im}(z)$ is a cont. function

$$\Rightarrow \mathrm{Im}\left(\frac{z - z_1}{z - z_2} \Big/ \frac{z_3 - z_1}{z_3 - z_2}\right) \text{ is cont at } z_*.$$

if $\operatorname{Im}\left(\frac{z_* - z_1}{z_* - z_2} \Big/ \frac{z_3 - z_1}{z_3 - z_2}\right) > 0$ and $\operatorname{Im}\left(\frac{w_* - z_1}{w_* - z_2} \Big/ \frac{z_3 - z_1}{z_3 - z_2}\right) < 0$.

and suppose the $\gamma(t) = \gamma_1(t) + i\gamma_2(t)$ is a cont path outside the circle. then $\operatorname{Im}\left(\frac{\gamma(t) - z_1}{\gamma(t) - z_2} \Big/ \frac{z_3 - z_1}{z_3 - z_2}\right)$ is a cont function in terms of t. Therefore by mean value Thm. there is a Q_* on the path s.t $\operatorname{Im}\left(\frac{Q_* - z_1}{Q_* - z_2} \Big/ \frac{z_3 - z_1}{z_3 - z_2}\right) = 0$.

But this is impossible. Since all such pt should be on the circle. this shows that $\operatorname{Im}\left(\frac{z - z_1}{z - z_2} \Big/ \frac{z_3 - z_1}{z_3 - z_2}\right) > 0$ outside the circle.

Derivatives

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

ex: $f(z) = \frac{1}{z}$ $\Rightarrow$ $f'(z) = -\frac{1}{z^2}$

ex: $f(z) = \bar{z}$. no derivative.

ex: $f(z) = |z|^2$. no derivative at $z \neq 0$.

Only derivable at $z = 0$.

Rk 1: $|z|^2 = (x^2+y^2)$

$\Rightarrow u(x,y) = x^2+y^2$. $v(x,y) = 0$.

$\therefore$ u, v are all smooth function. but $|z|^2$ is still not derivable in complex sense.

$\therefore$ $f(z) = u+iv$ derivable $\Rightarrow$ u, v derivable
$\Leftarrow\!\!\!\!\times$.

Rk 2: $\bar{z}$ is continuous. but not derivable.

$\therefore$ f is derivable $\Rightarrow$ f is continuous
$\Leftarrow\!\!\!\!\times$.

Derivative rules.

$c' = 0$. $z' = 1$. $(cf)' = cf'$. $(z^n)' = nz^{n-1}$

$(f \pm g)' = f' \pm g'$. $(fg)' = f'g + fg'$. $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$.

Cauchy Riemann

$f = u+iv$ is derivable

then
$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}.$$