THE CHINESE UNIVERSITY OF HONG KONG
MATH2230 Tutorial 9
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0.1 Differentiation and Integration of Larrent Series

Theorem 1. *Suppose that f is analytic in an annulus $A = \{z \in \mathbb{C} \mid r < |z - z_0| < R\}$. For any compact subset K of A, the Larrent series of f converges to f uniformly and absolutely for all $z \in K$.*

Theorem 2. *Suppose that f is analytic in an annulus $A = \{z \in \mathbb{C} \mid r < |z - z_0| < R\}$. For any $a \in A$, we can differentiate the Larrent series of f term by term. That is,*

$$f'(a) = \sum_{n=1}^{\infty} na_n(a - z_0)^{n-1} - \sum_{n=1}^{\infty} \frac{nb_n}{(a - z_0)^{n+1}}$$

Theorem 3. *Suppose that f is analytic in an annulus $A = \{z \in \mathbb{C} \mid r < |z - z_0| < R\}$. For any contour C inside A, we can integrate the Larrent series of f term by term. That is,*

$$\int_C f(z)dz = \sum_{n=0}^{\infty} a_n \int_C (z - z_0)^n dz + \sum_{n=1}^{\infty} b_n \int_C \frac{1}{(z - z_0)^n} dz$$

Remark : Theorem 2 and 3 are a immediate consequence of theorem 1.
Be careful that the contour in the above theorem may not be closed! If the contour is closed and contains z_0, we see that all the term are zero except the term $b_1 \int_C \frac{1}{(z-z_0)} dz$, it is because the terms $(z - z_0)^n$ have antiderivative in A except $\frac{1}{(z-z_0)}$ $(n = -1)$. This leads to an important theorem. Before that, we introduce some definitions.

0.2 Three Types of Isolated Singularity

There are three types of isolated singularity. We suppose that f is analytic function in $B_R(a) \setminus \{a\}$.(hence a is isolated singularity)

Definition 1. *The point a is called a removable singularity if there is an analytic function $\widetilde{f}$ in $B_R(a)$ such that $\widetilde{f} = f$ in $B_R(a) \setminus \{a\}$ ($\widetilde{f} = f$ except at $z = a$).*

Remark : It is the best behaved singularity, it is 'almost' an analytic function. From the definition, the singularity is removed by defining $\widetilde{f}$.

Theorem 4. *The point a is a removable singularity iff $\lim_{z \to a}(z - a)f(z) = 0$.*

Definition 2. *The point a is called a pole if $\lim_{z \to a} |f(z)| = \infty$.*

Theorem 5. *If f has a pole at $z = a$, then there is a positive integer m and an analytic function g in $B_R(a)$ with $g(a) \neq 0$ such that $f = \dfrac{g}{(z - a)^m}$. This m is called the order of pole of f at $z = a$.*

Definition 3. *The point a is called an essential singularity if it is not neither removable singularity nor pole.*

Remark : In this definition, we can see that $\lim_{z \to a} |f(z)|$ fails to exist, it will converges to different finite value and ∞ according to different path taken.

Theorem 6. *(Casorati-Weierstrass theorem) If f has essential singularity at $z = a$, then for every $c \in \mathbb{C}$, there is a sequence z_n converges to a such that $|f(z_n) - c| \to 0$.*

Remark : It tells us that given any $c \in \mathbb{C}$, there is z arbitrary close to a such that $f(z)$ arbitrary close to c. In other words, $f(z)$ can take any complex value, with at most one exception value, near $z = a$. (by Great Picard's Theorem)

In the view of Larrent series, we have the following conclusion,

Theorem 7. *Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{m} \frac{b_n}{(z-a)^n}$ be its Larrent series in $B_R(a) \setminus \{a\}$, then*

- *$z = a$ is a removable singularity iff $b_n = 0$ for $n \geq 1$,*
- *$z = a$ is a pole of order m iff $b_m \neq 0$ and $b_n = 0$ for $n \geq m+1$*
- *$z = a$ is an essential singularity iff $b_n \neq 0$ for infinitely many integers $n \geq 1$. (not necessary every n!)*

Remark : This theorem comes immediately from Theorem 4 and 5.

0.3 Residue Theory

Definition 4. *Suppose that f is analytic in some punctured disk $D = \{z \in \mathbb{C} \mid 0 < |z - z_0| < R\}$. The coefficient of $\frac{1}{(z-z_0)}$ in the Larrent series is called the residue of f at the singular point $z = z_0$, which is denoted by $\operatorname*{Res}_{z=z_0} f$. If we write $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$, then $\operatorname*{Res}_{z=z_0} f = b_1$.*

Theorem 8. *(Cauchy Residue Theorem) Suppose C is a closed contour in positive sense. If f is analytic inside and on C except finite number of singular points z_k inside C, then*

$$\int_C f dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f$$

Remark : Actually it is exactly Cauchy integral formula in the view of power series.
Remark : In other words, to calculate the integral $\int_C f dz$ is to calculate the residue of f at the singular points.

0.4 Exercise:

1. Compute $\displaystyle\int_C e^{-\frac{1}{z}} dz$ where C representing the contour $\{|z| = 3\}$.
2. Compute $\displaystyle\int_C \frac{\pi}{z^2 \sin(\pi z)} dz$ where C representing the contour $\left\{|z| = \frac{1}{2}\right\}$.