HW 10

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5(a) $\int_C \frac{dz}{z^3(z+4)} = 2\pi i \, Res\left(\frac{1}{z^3(z+4)}, 0\right)$

Around $z=0$, $\frac{1}{z^3(z+4)} = \frac{1}{4z^3} \frac{1}{1-(-\frac{z}{4})} = \frac{1}{4z^3} \sum_{n=0}^{\infty} \left(-\frac{z}{4}\right)^n = \sum_{n=-3}^{\infty} (-1)^{n+1} \frac{z^n}{4^{n+4}}$

So $\int_C \frac{dz}{z^3(z+4)} = 2\pi i \cdot \frac{1}{4^3} = \frac{\pi i}{32}$

(b) $\int_C \frac{dz}{z^3(z+4)} = 2\pi i \left(Res\left(\frac{1}{z^3(z+4)}, 0\right) + Res\left(\frac{1}{z^3(z+4)}, -4\right)\right)$

Around $z=-4$, $\frac{1}{z^3}$ is analytic.

$z=-4$ is a pole of order 1, which implies

$Res\left(\frac{1}{z^3(z+4)}, -4\right) = \frac{1}{(-4)^3} = -\frac{1}{64}$

So $\int_C \frac{dz}{z^3(z+4)} = 0$

6 let $f(z) = \frac{\cosh(\pi z)}{z(z^2+1)}$

$\int_C f \, dz = 2\pi i \left(Res(f,0) + Res(f,i) + Res(f,-i) \right)$

$= 2\pi i \left(\frac{\cosh(0)}{1} + \frac{\cosh(\pi i)}{i(i+i)} + \frac{\cosh(-\pi i)}{-i(-i-i)} \right)$

$= 2\pi i \left(1 - \frac{1}{2}(-1) - \frac{1}{2}(-1) \right) = 4\pi i$

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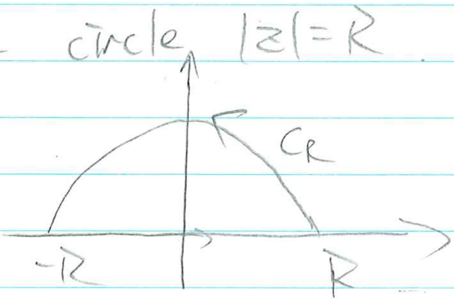
5. Let C_R be the upper half of the circle $|z|=R$.

Then for large enough R,



$\int_{-R}^{R} \frac{x^2}{(x^2+1)(x^2+4)} dx + \int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)} dz$

$= 2\pi i \left(Res(f,i) + Res(f,2i) \right)$, where $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$

$= 2\pi i \left(\frac{i^2}{(i+i)(i^2+4)} + \frac{(2i)^2}{((2i)^2+1)(2i+2i)} \right)$

$= 2\pi i \left(-\frac{1}{6i} + \frac{1}{3i} \right) = \frac{\pi}{3}$

Note that $\left| \int_{C_R} \frac{z^2}{(z^2+1)(z^2+4)} dz \right| \le \pi R \cdot \frac{R^2}{(R^2-1)(R^2-4)} \to 0$ as $R \to +\infty$.

and $f(x)$ is an even function. By take $R \to \infty$, we have

$\int_0^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx = \frac{\pi}{6}$

7. $x^2+2x+2 = (x-(-1+i))(x-(-1-i))$

let C_R as before, we have

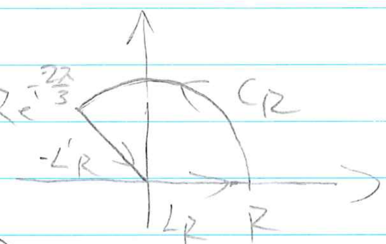
$$\int_{-R}^{R} \frac{1}{x^2+2x+2}\,dx + \int_{C_R} \frac{dz}{z^2+2z+2} = 2\pi i\left(\mathrm{Res}\left(\frac{1}{z^2+2z+2}, -1+i\right)\right)$$
$$= 2\pi i\left(\frac{1}{-1+i-(-1-i)}\right)$$
$$= \pi$$

$$\left|\int_{C_R} \frac{dz}{z^2+2z+2}\right| \le \pi R \frac{1}{(R-\sqrt{2})^2} \to 0 \quad \text{as} \quad R \to +\infty$$

Taking $R \to +\infty$, $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+2x+2}\,dx = \pi$

9. On L'_R, $z = re^{i\frac{2\pi}{3}}$, $r \in [0, R]$

$$\left(\int_{L_R} + \int_{C_R} - \int_{L'_R}\right) \frac{1}{z^3+1}\,dz = 2\pi i\,\mathrm{Res}\left(\frac{1}{z^3+1}, e^{i\frac{\pi}{3}}\right)$$



$$\left|\int_{C_R} \frac{1}{z^3+1}\,dz\right| \le \pi R \frac{1}{R^3-1} \to 0 \quad \text{as} \quad R \to +\infty$$

$$\int_{L'_R} \frac{1}{z^3+1}\,dz = \int_0^R \frac{e^{i\frac{2\pi}{3}}}{(re^{i\frac{2\pi}{3}})^3+1}\,dr = e^{i\frac{2\pi}{3}}\int_0^r \frac{1}{r^3+1}\,dr = e^{i\frac{2\pi}{3}}\int_{L_R} \frac{1}{z^3+1}\,dz$$

$$\mathrm{Res}\left(\frac{1}{z^3+1}, e^{i\frac{\pi}{3}}\right) = \lim_{z \to e^{i\frac{\pi}{3}}} \frac{z-e^{i\frac{\pi}{3}}}{z^3+1} \overset{\text{L'Hospital}}{=} \lim_{z \to e^{i\frac{\pi}{3}}} \frac{1}{3z^2} = \frac{1}{3e^{i\frac{2\pi}{3}}}$$

Taking $R \to +\infty$, we have

$$\left(1-e^{i\frac{2\pi}{3}}\right)\int_0^\infty \frac{dx}{x^3+1} = 2\pi i\left(\frac{1}{3e^{i\frac{2\pi}{3}}}\right)$$

$$\int_0^\infty \frac{dx}{x^3+1} = \frac{2\pi i}{3}\frac{1}{(1-e^{i\frac{2\pi}{3}})e^{i\frac{2\pi}{3}}} = \frac{\pi}{3}\frac{1}{\frac{1}{2i}(e^{i\frac{\pi}{3}}-e^{-i\frac{\pi}{3}})(e^{i\frac{3\pi}{3}})} = \frac{\pi}{3}\frac{1}{\sin\frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}}$$

10 (a) $z^{2n}+1=0 \Rightarrow z^{2n} = e^{i\pi} \Rightarrow z = \exp\left(i\frac{\pi+2k\pi}{2n}\right)$, $k=0,\ldots,2n-1$

Note that $\frac{\pi+2k\pi}{2n} < \pi$ for $k=0,1,\ldots,n-1$

Also, for $k=n,\ldots,2n-1$, $\frac{\pi+2k\pi}{2n} > \pi$

So for $k=0,\ldots,n-1$, c_k lies above the x-axis and for $k=n,\ldots,2n-1$, c_k lies below the real axis. There are none on that axis

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10(b) $\underset{z=c_k}{\mathrm{Res}} \frac{z^{2m}}{z^{2n}+1} = \frac{c_k^{2m}}{2n c_k^{2n-1}} = \frac{c_k^{2m+1}}{2n c_k^{2n}} = -\frac{1}{2n}\exp\left(i\frac{(2k+1)\pi}{2n}\cdot(2m+1)\right)$

$= -\frac{1}{2n} e^{i(2k+1)\alpha}$

$$2\pi i \sum_{k=0}^{n-1} \underset{z=c_k}{\mathrm{Res}} \frac{z^{2m}}{z^{2n}+1} = 2\pi i \sum_{n=0}^{n-1} -\frac{1}{2n} e^{i(2k+1)\alpha}$$

$$= -\frac{\pi i}{n} e^{i\alpha} \frac{1-e^{i2n\alpha}}{1-e^{i2\alpha}}$$

$$= \frac{\pi}{n} \frac{1}{\frac{e^{i\alpha}-e^{-i\alpha}}{2i}} \qquad \left(e^{i2n\alpha} = -1\right)$$

$$= \frac{\pi}{n\sin\alpha}$$

(c) The argument is the same as the previous questions.

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2. Let $f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$

Since i is a simple pole of $f(z)e^{iaz}$, write $f(z)e^{iaz} = \frac{1}{z-i}\phi(z)$, where $\phi(z) = \frac{e^{iaz}}{z+i}$ is analytic on the upper half plane

$$\underset{z=i}{\mathrm{Res}} f(z)e^{iaz} = \frac{e^{iai}}{i+i} = \frac{e^{-a}}{2i}$$

So $$\int_{-R}^{R} \frac{e^{iax}}{x^2+1}dx + \int_{C_R} \frac{e^{iaz}}{z^2+1}dz = 2\pi i\left(\frac{e^{-a}}{2i}\right) = \pi e^{-a} \qquad (*)$$

with C_R the upper half circle centre at 0 with radius R.

On C_R, since $\left|\frac{1}{z^2+1}\right| \le \frac{1}{R^2-1} \to 0$ as $R \to +\infty$,

by Jordan's lemma.

$$\left|\mathrm{Re}\int_{C_R} \frac{e^{iaz}}{z^2+1}dz\right| \le \left|\int_{C_R} \frac{e^{iaz}}{z^2+1}dz\right| \to 0 \text{ as } R \to +\infty$$

By comparing the real part of (*),

$$\int_0^\infty \frac{\cos ax}{x^2+1}dx = \lim_{R\to+\infty} \frac{1}{2}\int_{-R}^{R} \frac{\cos ax}{x^2+1} = \frac{\pi}{2}e^{-a}$$

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5. Let $f(z) = \dfrac{z^3}{z^4+4}$

Poles of $f(z)e^{iaz}$ on the upper half plane: $\sqrt{2}e^{i\frac{\pi}{4}}$, $\sqrt{2}e^{i\frac{3\pi}{4}}$

$$\operatorname*{Res}_{z=\sqrt{2}e^{i\frac{\pi}{4}}} f(z)e^{iaz} = \frac{(\sqrt{2}e^{i\frac{\pi}{4}})^3 e^{ia(\sqrt{2}e^{i\frac{\pi}{4}})}}{4(\sqrt{2}e^{i\frac{\pi}{4}})^3} \quad \text{(See P.265 10 (b))}$$

$$= \frac{1}{4}\exp\left(ia\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)\right)$$

$$= \frac{1}{4}\exp(-a + ai)$$

$$= \frac{1}{4}e^{-a}e^{ai}$$

Similarly, $\displaystyle\operatorname*{Res}_{z=\sqrt{2}e^{i\frac{3\pi}{4}}} f(z)e^{iaz} = \frac{1}{4}\exp\left(ia\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)\right)$

$$= \frac{1}{4}e^{-a}e^{-ai}$$

So $$\int_{-R}^{R} \frac{x^3e^{iax}}{x^4+4}dx + \int_{C_R}\frac{z^3e^{iaz}}{z^4+4}dz = 2\pi i\left(\frac{1}{4}e^{-a}e^{ai} + \frac{1}{4}e^{-a}e^{-ai}\right)$$

$$= i\pi e^{-a}\cos a \qquad (*)$$

with C_R same as Q2.

Since on C_R, $\left|\dfrac{z^3}{z^4+4}\right| \le \dfrac{R}{R^4-4} \to 0$ as $R \to +\infty$,

By Jordan's lemma,

$$\left|\operatorname{Im}\int_{C_R}\frac{z^3e^{iaz}}{z^4+4}dz\right| \le \left|\int_{C_R}\frac{z^3e^{iaz}}{z^4+4}dz\right| \to 0 \text{ as } R\to+\infty.$$

By comparing the imaginary part of $(*)$,

$$\int_{-\infty}^{\infty}\frac{x^3\sin ax}{x^4+4}dx = \pi e^{-a}\cos a$$

7. Let $f(z) = \dfrac{z^3}{(z^2+1)(z^2+9)}$

Poles of $f(z)e^{iz}$ on the upper half plane: i and $3i$.

$$\operatorname*{Res}_{z=i} f(z)e^{iz} = \frac{i^3e^{i\cdot i}}{2i(i^2+9)} = \frac{-ie^{-1}}{2i(8)} = -\frac{e^{-1}}{16}$$

$$\operatorname*{Res}_{z=3i} f(z)e^{iz} = \frac{(3i)^3e^{i\cdot 3i}}{((3i)^2+1)\cdot 6i} = \frac{-27ie^{-3}}{(-8)\cdot 6i} = \frac{9e^{-3}}{16}$$

So $\displaystyle\int_{-R}^{R}\frac{x^3e^{ix}}{(x^2+1)(x^2+9)}dx + \int_{C_R}\frac{z^3e^{iz}}{(z^2+1)(z^2+9)}dz = 2\pi i\left(-\frac{e^{-1}}{16} + \frac{9e^{-3}}{16}\right)$, C_R as Q5

Check that on C_R, $\left|\dfrac{z^3}{(z^2+1)(z^2+9)}\right| \le \dfrac{R^3}{(R^2-1)(R^2-9)} \to 0$ as $R\to+\infty$

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7. Using the fact that $f(x)$ is even, and the exactly the same argument in Q5,

$$\int_0^{\infty} \frac{x^3 \sin x}{(x^2+1)(x^2+9)}\,dx = \pi\left(-\frac{e^{-1}}{16} + \frac{9e^{-3}}{16}\right)$$

8. Let $f(z) = \dfrac{1}{z^2+4z+5} = \dfrac{1}{(z-(-2+i))\,(z-(-2-i))}$

Pole of $f(z)e^{iz}$ on the upper half plane : $-2+i$

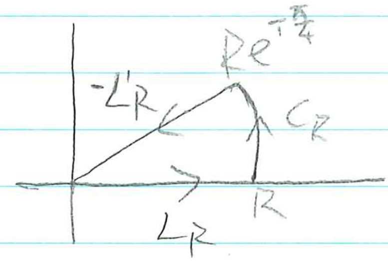
$$\operatorname*{Res}_{z=-2+i} f(z)e^{iz} = \frac{e^{i(-2+i)}}{-2+i-(-2-i)} = \frac{e^{-1-2i}}{2i} = \frac{e^{-1}}{2i}(\cos 2 - i\sin 2)$$

So $$\int_{-R}^{R} \frac{e^{ix}}{x^2+4x+5}\,dx + \int_{C_R} \frac{e^{iz}}{z^2+4z+5}\,dz = \pi e^{-1}(\cos 2 - i\sin 2)$$

On C_R, $\left|\dfrac{1}{z^2+4z+5}\right| \le \dfrac{1}{(R-\sqrt{5})^2} \to 0$ as $R \to +\infty$.

Argue as above, $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\,dx = \frac{\pi}{e}\sin 2$

12(a) On L'_R, $z = re^{i\frac{\pi}{4}}$, $r \in [0, R]$



$$\int_{L'_R} \exp(iz^2)\,dz = \int_0^R \exp(ir^2 e^{i\frac{\pi}{2}})e^{i\frac{\pi}{4}}\,dr$$
$$= \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\int_0^r e^{-r^2}\,dr$$

Since $\exp(iz^2)$ is analytic inside the bdd domain, by Cauchy-Goursat thm,

$$\int_0^R \exp(ix^2)\,dx - \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\int_0^r e^{-r^2}\,dr + \int_{C_R} \exp(iz^2)\,dz = 0.$$

By comparing the real part and imaginary parts of above, the result follows.

(b) $$\left|\int_{C_R} e^{iz^2}\,dz\right| = \left|\int_0^{\frac{\pi}{4}} e^{iR^2(\cos 2\theta + i\sin 2\theta)}\, iRe^{i\theta}\,d\theta\right|$$
$$\le R\int_0^{\frac{\pi}{4}} \left|e^{-R^2\sin 2\theta}\right|\,d\theta$$
$$= \frac{R}{2}\int_0^{\frac{\pi}{2}} e^{-R^2\sin\phi}\,d\phi, \quad \phi = 2\theta$$
$$\le \frac{R}{2}\cdot\frac{\pi}{2R^2} = \frac{\pi}{4R} \to 0 \quad \text{as } R \to +\infty$$

by the form (2), Sec 88.

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(c) $\int_0^\infty \cos(x^2)\,dx = \lim_{R\to\infty}\left(\frac{1}{\sqrt{2}}\int_0^R e^{-r^2}\,dr - \mathrm{Re}\int_{C_R} e^{iz^2}\,dz\right)$

$= \frac{1}{\sqrt{2}}\int_0^\infty e^{-r^2}\,dr$

$= \frac{1}{\sqrt{2}}\cdot\sqrt{\frac{\pi}{2}}$

$= \frac{1}{2}\sqrt{\frac{\pi}{2}}$, since $\left|\mathrm{Re}\int_{C_R} e^{iz^2}\,dz\right| \le \left|\int_{C_R} e^{iz^2}\,dz\right| \to 0$ as $R\to+\infty$.

Similar for $\int_0^\infty \sin(x^2)\,dx$