THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
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Tutorial 9
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0.1 Differentiation and Integration of Larrent Series

Theorem 1. *Suppose that f is analytic in an annulus $A = \{z \in \mathbb{C} \mid r < |z - z_0| < R\}$. For any compact subset K of A, the Larrent series of f converges to f uniformly and absolutely for all $z \in K$*

Theorem 2. *Suppose that f is analytic in an annulus $A = \{z \in \mathbb{C} \mid r < |z - z_0| < R\}$. For any $a \in A$, we can differentiate the Larrent series of f term by term. That is,*

$$f'(a) = \sum_{n=1}^{\infty} na_n(a - z_0)^{n-1} - \sum_{n=1}^{\infty} \frac{nb_n}{(a - z_0)^{n+1}}$$

Theorem 3. *Suppose that f is analytic in an annulus $A = \{z \in \mathbb{C} \mid r < |z - z_0| < R\}$. For any contour C inside A, we can integrate the Larrent series of f term by term. That is,*

$$\int_C f(z)dz = \sum_{n=0}^{\infty} a_n \int_C (z - z_0)^n dz + \sum_{n=1}^{\infty} b_n \int_C \frac{1}{(z - z_0)^n} dz$$

Remark : Theorem 2 and 3 are a immediate consequence of theorem 1.
Be careful that the contour in the above theorem may not be closed! If the contour is closed and contains z_0, we see that all the term are zero except the term $b_1 \int_C \frac{1}{(z-z_0)} dz$, it is because the terms $(z - z_0)^n$ have antiderivative in A except $\frac{1}{(z-z_0)}$ $(n = -1)$. This leads to an important theorem. Before that, we introduce some definitions.

0.2 Three Types of Isolated Singularity

There are three types of isolated singularity. We suppose that f is analytic function in $B_R(a) \setminus \{a\}$.(hence a is isolated singularity)

Definition 1. *The point a is called a removable singularity if there is an analytic function $\widetilde{f}$ in $B_R(a)$ such that $\widetilde{f} = f$ in $B_R(a) \setminus \{a\}$ ($\widetilde{f} = f$ except at $z = a$).*

Remark : It is the best behaved singularity, it is 'almost' an analytic function. From the definition, the singularity is removed by defining $\widetilde{f}$.

Theorem 4. *The point a is a removable singularity iff $\lim_{z \to a}(z - a)f(z) = 0$.*

Definition 2. *The point a is called a pole if $\lim_{z \to a} |f(z)| = \infty$.*

Theorem 5. *If f has a pole at $z = a$, there is a positive integer m and an analytic function g in $B_R(a)$ such that $f = \dfrac{g}{(z - a)^m}$. This m is called the order of pole of f at $z = a$.*

Definition 3. *The point a is called an essential singularity if it is not neither removable singularity nor pole.*

Remark : In this definition, we can see that $\lim_{z \to a} |f(z)|$ fails to exist, it will converges to different finite value and ∞ according to different path taken.

Theorem 6. *(Casorati-Weierstrass theorem) If f has essential singularity at $z = a$, then for every $c \in \mathbb{C}$, there is a sequence z_n converges to a such that $|f(z_n) - c| \to 0$.*

Remark : It tells us that given any $c \in \mathbb{C}$, there is z arbitrary close to a such that $f(z)$ arbitrary close to c. In other words, $f(z)$ can take any complex value near $z = a$.

In the view of Larrent series, we have the following conclusion,

Theorem 7. *Let $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{m} \dfrac{b_n}{(z-a)^n}$ be its Larrent series in $B_R(a) \setminus \{a\}$, then*

- *$z = a$ is a removable singularity iff $b_n = 0$ for $n \geq 1$,*
- *$z = a$ is a pole of order m iff $b_m \neq 0$ and $b_n = 0$ for $n \geq m+1$*
- *$z = a$ is an essential singularity iff $b_n \neq 0$ for infinitely many integers $n \geq 1$. (not necessary every n!)*

Remark : This theorem comes immediately from Theorem 4 and 5.

0.3 Residue Theory

Definition 4. *Suppose that f is analytic in some punctured disk $D = \{z \in \mathbb{C} \mid 0 < |z - z_0| < R\}$. The coefficient of $\dfrac{1}{(z-z_0)}$ in the Larrent series is called the residue of f at the singular point $z = z_0$, which is denoted by $\operatorname*{Res}_{z=z_0} f$. If we write $f(z) = \sum\limits_{n=0}^{\infty} a_n(z-z_0)^n + \sum\limits_{n=1}^{\infty} \dfrac{b_n}{(z-z_0)^n}$, then $\operatorname*{Res}_{z=z_0} f = b_1$.*

Theorem 8. *(Cauchy Residue Theorem) Suppose C is a closed contour in positive sense. If f is analytic inside and on C except finite number of singular points z_k inside C, then*

$$\int_C f dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f$$

Remark : Actually it is exactly Cauchy integral formula in the view of power series.
Remark : In other words, to calculate the integral $\int_C f dz$ is to calculate the residue of f at the singular points.

Then we come to the computation of residue. Of course we can express the whole Larrent series to obtain that. We provide an alternative method here. If the order of pole of f at $z = z_0$ is m and thus

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{m} \frac{b_n}{(z-z_0)^n}.$$

We consider

$$(z-z_0)^m f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^{n+m} + b_1(z-z_0)^{m-1} + b_2(z-z_0)^{m-2} + \ldots + b_m$$

and differentiate it $m-1$ times, we could have

$$\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)] = (m-1)!b_1 + O(z-z_0)$$

Theorem 9. *Suppose that f is analytic in some punctured disk $D = \{z \in \mathbb{C} \mid 0 < |z-z_0| < R\}$ and the order of pole at z_0 is m, then* $\operatorname*{Res}_{z=z_k} f = \lim_{z\to z_0} \frac{1}{(m-1)!}\left(\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\right)$.

0.4 Exercise:

1. Compute $\int_C e^{-\frac{1}{z}} dz$ where C representing the contour $\{|z| < 3\}$.

2. Compute $\int_C \frac{\pi}{z^2 \sin(\pi z)} dz$ where C representing the contour $\{|z| < \frac{1}{2}\}$.

3. Compute $\int_0^{\pi/2} \frac{d\theta}{a^2 + \sin^2\theta}$ for $a > 0$.