MATH 2230B HW10 Solution

(1a) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \Rightarrow e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n! e^z}$

Therefore $z e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n! z^{n-1}} = 1 + z + \sum_{n=1}^{\infty} \frac{1}{(n+1)! z^n}$

It is essential singularity.

(1b) $\frac{z^2}{z+1} = \frac{(z+1)^2 - 2z - 1}{z+1} = \frac{(z+1)^2 - 2(z+1) + 1}{z+1}$

$= (z+1) - 2 + \frac{1}{z+1}$

It is pole of order 1.

(1c) $\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \Rightarrow \frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}$

It is removable singularity.

(1d) $\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \Rightarrow \frac{\cos z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n-1}}{(2n)!}$

$= \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n-1}}{(2n)!} + \frac{1}{z}$

It is pole of order 1.

(1e) It is clear that $z=2$ is a pole of order 3.

(2a) $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \Rightarrow \frac{1 - \cosh z}{z^3} = \sum_{n=1}^{\infty} - \frac{z^{2n-3}}{(2n)!}$

$= -\frac{1}{2z} - \sum_{n=2}^{\infty} \frac{z^{2n-3}}{(2n)!}$

It is pole of order 1 with residue $-\frac{1}{2}$.

(2b) $e^{2z} = \sum_{n=0}^{\infty} \frac{(2z)^n}{n!} \Rightarrow \frac{1-e^{2z}}{z^4} = -\sum_{n=1}^{\infty} \frac{2^n z^{n-4}}{n!}$

$= -\sum_{n=4}^{\infty} \frac{2^n z^{n-4}}{n!} - \frac{2}{z^3} - \frac{2}{z^2} - \frac{4}{3z}$

It is pole of order 3 with residue $-4/3$.

(2c) $e^{2(z-1)} = \sum_{n=0}^{\infty} \frac{(2(z-1))^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!}(z-1)^n$

$\Rightarrow e^{2z} = e^2 \sum_{n=0}^{\infty} \frac{2^n}{n!}(z-1)^n$

$\frac{e^{2z}}{(z-1)^2} = \sum_{n=0}^{\infty} \frac{e^2 2^n}{n!}(z-1)^{n-2} = \sum_{n=2}^{\infty} \frac{e^2 2^n}{n!}(z-1)^{n-2} + \frac{e^2}{(z-1)^2} + \frac{2e^2}{z-1}$

It is pole of order 2 with residue $2e^2$.

(3a) Method 1: Since f is analytic,

$f = \sum_{n=0}^{\infty} a_n(z-z_0)^n \Rightarrow g = \frac{f}{z-z_0} = \sum_{n=0}^{\infty} a_n(z-z_0)^{n-1}$

Since $f(z_0) = a_0 \neq 0 \Rightarrow z_0$ is simple pole with residue $f(z_0)$

Method 2: We see that $\lim_{z\to z_0} g\cdot(z-z_0) = \lim_{z\to z_0} f(z) = f(z_0)$ as f is analytic.

And $\lim_{z\to z_0} g\cdot(z-z_0)^2 = \lim_{z\to z_0} f\cdot(z-z_0) = 0$ as f is analytic.

Thus, $n=2$ is the least integer satisfies $\lim_{z\to z_0} g\cdot(z-z_0)^n = 0$

Results follow.

(3b) Method 1: From 3a, $g = \sum_{n\geq 0}^{\infty} a_n (z-z_0)^{n-1}$

$$= \sum_{n=1}^{\infty} a_n (z-z_0)^{n-1} \quad (a_0 = f(z_0) = 0)$$

Therefore, z_0 is removable singularity.

Method 2: $\lim_{z\to z_0} g\cdot(z-z_0) = f(z_0) = 0 \Rightarrow$ Result follows.

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(3) a $\sinh z = \sum_{n\geq 0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \Rightarrow \frac{\sinh z}{z^4} = \sum_{n\geq 0}^{\infty} \frac{z^{2n-3}}{(2n+1)!}$

It is pole of order 3 with residue 1/6.

(3b) $e^z - 1 = \sum_{n\geq 1}^{\infty} \frac{z^n}{n!} \Rightarrow \frac{1}{z(e^z-1)} = \frac{1}{\sum_{n\geq 1}^{\infty} \frac{z^{n+1}}{n!}} = \frac{1}{z^2\left(1 + \sum_{n\geq 2}^{\infty} \frac{z^{n-1}}{n!}\right)}$

It is pole of order 2.

$$\operatorname*{Res}_{z=0} \frac{1}{z(e^z-1)} = \lim_{z\to 0} \frac{d}{dz}\left(\frac{z^2}{z(e^z-1)}\right)$$

$$= \lim_{z\to 0}\left(\frac{(e^z-1) - ze^z}{(e^z-1)^2}\right)$$

$$= \lim_{z\to 0} \frac{e^z - e^z - ze^z}{2(e^z-1)e^z}$$

$$= \lim_{z\to 0} \frac{-z}{2(e^z-1)}$$

$$= \lim_{z\to 0} \frac{-1}{2e^z} = -\frac{1}{2}.$$

(5a) It is enough to find the residue of $\frac{1}{z^3(z+4)}$ at $z=0$.

$$\frac{1}{z+4} = \frac{1}{4\left(1+\frac{z}{4}\right)} = \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{z}{4}\right)^n \quad \left(\left|\frac{z}{4}\right| = \frac{1}{2} < 1 \text{ in } \{|z|=2\}\right)$$

$$\frac{1}{z^3(z+4)} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{z^{n-3}}{(-4)^n} \Rightarrow \operatorname*{Res}_{z=0}\frac{1}{z^3(z+4)} = \frac{1}{64}$$

$$\therefore \int_C \frac{dz}{z^3(z+4)} = \frac{\pi i}{32}$$

(5b) It is enough to find the residue of $\frac{1}{z^3(z+4)}$ at $z=0$ and $z=-4$.

From (a), $\operatorname*{Res}_{z=0}\frac{1}{z^3(z+4)} = \frac{1}{64}$

$$\cancel{\frac{1}{z} = \frac{1}{-4+z+4} = \frac{-1}{4}\left(\frac{1}{1-\frac{(z+4)}{4}}\right) = -\frac{1}{4}\left(\sum_{n=0}^{\infty}\left(\frac{z+4}{4}\right)^n\right)}$$

$$\operatorname*{Res}_{z=-4}\frac{1}{z^3(z+4)} = \lim_{z\to -4}\frac{1}{z^3(z+4)}(z+4) = \frac{1}{-64}$$

$$\therefore \int_C \frac{dz}{z^3(z+4)} = 0$$

(6) $$\frac{\cosh \pi z}{z(z^2+1)} = \frac{\cosh \pi z}{z(z+i)(z-i)}$$

$$\operatorname*{Res}_{z=0}\frac{\cosh\pi z}{z(z^2+1)} = \cancel{\operatorname*{Res}_{z=0}}\lim_{z\to 0}\frac{\cosh \pi z}{z^2+1} = 1$$

$$\operatorname*{Res}_{z=i}\frac{\cosh\pi z}{z(z^2+1)} = \lim_{z\to i}\frac{\cosh\pi z}{z(z+i)} \cancel{\frac{e^{-\pi i}+e^{\pi i}}{2}} = \frac{1}{2}$$

$$\operatorname*{Res}_{z=-i} \frac{\cosh \pi z}{z(z^2+1)} = \lim_{z\to -i} \frac{\cosh \pi z}{z(z-i)} = \frac{1}{2}$$

$$\int_C \frac{\cosh \pi z\, dz}{z(z^2+1)} = 2\pi i (2) = 4\pi i$$

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5a Since $\tan z$ is singular at $z = \pm \pi/2$ in $\{|z|<2\}$.

$$\operatorname*{Res}_{z=\pi/2} \tan z = \lim_{z\to \pi/2} \frac{\sin z}{\cos z} \cdot (z - \pi/2)$$

$$= \lim_{z\to \pi/2} \frac{\cos z (z-\pi/2) + \sin z}{-\sin z}$$

$$\cancel{= \lim_{z\to \pi/2}} = -1$$

Similarly, $\operatorname*{Res}_{z=-\pi/2} \tan z = -1$

$$\int_C \tan z\, dz = 2\pi i(-2) = -4\pi i$$

5b $\cancel{\operatorname*{Res}_{z=0} \frac{1}{\sinh 2z} = \lim_{z\to 0} \frac{z}{\sinh 2z} = \lim_{z\to 0} \frac{1}{2\cosh 2z} = \frac{1}{2}}$

The zeros of $\sinh 2z$ in $\{|z|<2\}$ are $z = \pm \frac{in\pi}{2}$ $n=0, 1$ ~~in side $\{|z|<2\}$~~

$$\operatorname*{Res}_{z=\pm\frac{in\pi}{2}} \frac{1}{\sinh 2z} = \lim_{z\to \pm\frac{in\pi}{2}} \frac{z \pm in\pi/2}{\sinh 2z}$$

$$= \lim_{z\to \pm\frac{in\pi}{2}} \frac{1}{2\cosh 2z}$$

$$= \begin{cases} 1/2 & n=0 \\ -1/2 & n=1 \end{cases}$$

$$\therefore \int_C \frac{dz}{\sin 2z} = -\pi i$$

(6) The poles inside C_N are $0, \pm n\pi$, $n=1,2,\dots N$

The order of pole of $\frac{1}{z^2 \sin z}$ at zero is 3, then by long division, $\frac{1}{\sin z} = \frac{1}{z} + \frac{z}{6} + \dots$

$$\operatorname*{Res}_{z=0} \frac{1}{z^2 \sin z} = \frac{1}{6}.$$

$$\operatorname*{Res}_{z=\pm n\pi} \frac{1}{z^2 \sin z} = \lim_{z \to \pm n\pi} \frac{1}{z^2}\left(\frac{z \pm n\pi}{\sin z}\right) = \frac{(-1)^n}{n^2\pi^2}$$

Thus, $\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i\left(\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right)$

Since $\int_{C_N} \frac{dz}{z^2 \sin z} \to 0$ as $N \to \infty$,

$$2\pi i\left(\frac{1}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2}\right) = 0$$

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$