MATH5011

Exercise 5 Suggested Solution

Problems 2 and 3 are optional. In Problems 6, 7 and 8 we continue the study of the n-dimensional Lebesgue measure starting in Exercise 3. In Problems 9-11 we review the basic facts on the Cantor set. Both topics will be further discussed in Lecture 6.

(1) Let $f : X \to (-\infty, \infty]$ be l.s.c. (lower semi-continuous) where X is a topological space. Show that

(a) αf is l.s.c. $\forall \alpha \geq 0$,

(b) g l.s.c $\Rightarrow \min\{f, g\}$ l.s.c,

(c) f_α l.s.c $\Rightarrow \sup\limits_{\alpha} f_\alpha$ l.s.c.,

(d) g l.s.c. $\Rightarrow f + g$ l.s.c.

(e) $\infty > f > 0 \Rightarrow 1/f$ is u.s.c..

Solution:

(a) When $\alpha = 0$,

$$(\alpha f)^{-1}((t, \infty]) = \begin{cases} X, & \text{if } t < 0, \\ \phi, & \text{if } t \geq 0, \end{cases}$$

which is clearly open. When $\alpha > 0$, the assertion follows from

$$(\alpha f)^{-1}((t, \infty]) = \{x : \alpha f(x) > t\} = f^{-1}\big((t/\alpha, \infty]\big).$$

(b) It follows from

$$\begin{aligned} (\min\{f, g\})^{-1}((t, \infty]) &= \{x \in X : f(x) > t \text{ and } g(x) > t\} \\ &= f^{-1}((t, \infty]) \cap g^{-1}((t, \infty]). \end{aligned}$$

(c) It follows from

$$\left(\sup_\alpha f_\alpha\right)^{-1}(t,\infty] = \{x \in X : f_\alpha > t \text{ for some } \alpha\} = \bigcup_\alpha f^{-1}((t,\infty]).$$

(d) It follows from

$$\begin{aligned}(f+g)^{-1}(t,\infty] &= \{x \in X : f(x)+g(x) > t\}\\ &= \bigcup_{r\in\mathbb{Q}}\{x \in X : f(x) > r > t-g(x)\}\\ &= \bigcup_{r\in\mathbb{Q}}(f^{-1}((r,\infty]) \cap g^{-1}((t-r,\infty])).\end{aligned}$$

(e) It suffices to check that for any real number t, we have$\{x \in X : -\frac{1}{f(x)} > t\}$ is open. Case 1 (t is non-negative).

$$\{x \in X : -\frac{1}{f(x)} > t\} = \{x \in X : -1 > tf(x)\} = \phi.$$

Case 2 (t is negative).

$$\{x \in X : -\frac{1}{f(x)} > t\} = \{x \in X : -1 > tf(x)\} = \{x \in X : -\frac{1}{t} < f(x)\}$$

is open by lower semi-continuity of f.

(2) Let X be a locally compact Hausdorff space. Let $f \geq 0$ be l.s.c.. Show that

$$f = \sup\{g : g \in C_c(X),\ g \geq 0,\ g \leq f\}.$$

(Hint: Use Urysohn's lemma to construct, for $0 < a < f(x_0)$, $g(x_0) = a$, $g \in [0,a]$, etc.)

Solution: If $f(x_0) = 0$, then $g(x) \equiv 0$ satisfies condition. If $f(x_0) > 0$, for $\varepsilon > 0$, we need to find a l.s.c g, $f \geq g \geq 0$ and $g(x_0) \geq f(x_0) - \varepsilon$. By l.s.c, $\exists$ open set $G \ni x_0$ s.t.

$$f(x) > f(x_0) - \varepsilon, \forall x \in G.$$

Fix $G_1 \ni x_0$, open, $\overline{G_1}$ is compact and $\overline{G_1} \subset G$. Using Urysohn's lemma, $\exists \varphi$, $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $\overline{G_1}$, spt$\varphi \subset G$. Then $g(x) = (f(x_0) - \varepsilon)\varphi(x)$ satisfies $g(x) \leq f(x), \forall x \in G$ and

$$g(x_0) = (f(x_0) - \varepsilon)\varphi(x_0) = f(x_0) - \varepsilon \leq f(x_0).$$

(3) Let X be a compact topological space. Show that every l.s.c function from X to $\mathbb{R}$ attains its minimum, that is, there exists some $x \in X$ such that $f(x) \leq f(y)$, $\forall y \in X$.

Solution: Let $f : X \to \mathbb{R}$ l.s.c, X compact. First, we claim that $\exists$ m s.t.

$$f(x) \geq m, \forall x \in X.$$

$G_n = \{x : f(x) > n\}, n \in \mathbb{Z}$ is open. By compactness of X,

$$X = \bigcup_{n=1}^{\infty} G_n \Rightarrow X = \bigcup_{n=1}^{N} G_n.$$

Therefore $f(x) \geq N$. Second, let $\{x_n\}, f(x_n) \to \inf f \equiv m$. $F_j = \{x : f(x) \leq m + 1/j\}$ is non-empty closed set, so

$$\bigcap_{j=1}^{\infty} F_j \neq \phi$$

and

$$\exists x_0 \in \bigcap_{j=1}^{\infty} F_j, f(x_0) \leq m \Rightarrow f(x_0) = m.$$

(4) Show that every semicontinuous function is a Borel function.

Solution: Let f be a lower semicontinuous function. As every open set in $[-\infty, \infty]$ can be written as a countable union of (a, b), $[\infty, b)$, $(a, \infty]$ and f takes value in $(-\infty, \infty]$. Therefore it suffices to show that $f^{-1}(a, b)$ and $f^{-1}(a, \infty]$ are Borel sets. By the proposition 2.14, the later set is an open set and hence is a Borel set. $X \setminus f^{-1}(a, \infty] = f^{-1}(-\infty, a]$ is closed and

$$f^{-1}(-\infty, a) = \bigcup_{n=1}^{\infty} f^{-1}(-\infty, a - 1/n]$$

is a Borel set. Therefore

$$f^{-1}(a, b) = f^{-1}(-\infty, b) \cap f^{-1}(a, \infty]$$

is a Borel set. We have the preimage of open set of f is Borel set, so f is Borel function. the case for upper semicontinuos is similar.

(5) Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable. Show that there exist Borel measurable functions g, h, $g(x) \leq f(x) \leq h(x)$ for all $x \in \mathbb{R}^n$ such that $g(x) = h(x)$ a.e.

Solution: As Lebesgue measure on $\mathbb{R}^n$ is σ-finite and outer regular, we apply Proposition 2.17 of Chapter 2 and obtain a Borel function $\tilde{g}$ and a null set N s.t.

$$f(x) = \tilde{g}(x), \forall x \in \mathbb{R}^n \setminus N.$$

By outer regularity of Lebesgue measure, $\exists G_\delta$ set W s.t W is Lebesgue measure zero and $N \subseteq W$. We define two Borel functions in the following way,

$$g(x) = \begin{cases} \tilde{g}(x), & \text{if } x \in \mathbb{R}^n \setminus W, \\ -\infty, & \text{if } x \in W, \end{cases}$$

$$h(x) = \begin{cases} \tilde{g}(x), & \text{if } x \in \mathbb{R}^n \setminus W, \\ \infty, & \text{if } x \in W, \end{cases}$$

Obviously, g and h are Borel functions s.t

$$g(x) \leq f(x) \leq h(x), \forall x \in \mathbb{R}^n$$

and

$$g(x) = h(x) \text{ a.e..}$$

(6) Let λ be a Borel measure and μ a Riesz measure on $\mathbb{R}^n$ such that $\lambda(G) = \mu(G)$ for all open sets G. Show that λ coincides with μ on $\mathcal{B}$.

Solution: Let E be a Borel set. $\forall \varepsilon > 0, \exists$ closed set $F \subseteq E$ and open set $G \supseteq E$ s.t

$$\mu(G \setminus F) < \varepsilon,$$

so

$$\mu(G) \leq \mu(F) + \mu(G \setminus F) < \mu(E) + \varepsilon.$$

As $G \setminus F$ is open, $\lambda(G \setminus F) = \mu(G \setminus F) < \varepsilon$, so

$$\begin{aligned} \lambda(E) &\leq \lambda(G) \\ &= \mu(G) \\ &< \mu(E) + \varepsilon \Rightarrow \lambda(E) \leq \mu(E). \end{aligned}$$

$$\begin{aligned}\mu(E) \le \mu(G) = \lambda(G) &\le \lambda(F) + \lambda(G \setminus F) \\ &\le \lambda(E) + \varepsilon \Rightarrow \mu(E) \le \lambda(E).\end{aligned}$$

(7) A characterization of the Lebesgue measure based on translational invariance. Let $(\mathbb{R}^n, \mathcal{B}, \mu)$ be a Borel measure space whose measure μ is translational invariant and is nontrivial in the sense that there exists some Borel set A such that $\mu(A) \in (0, \infty)$. Show that there exists a positive constant c such that $c\mu$ is the restriction of the Lebesgue measure on $\mathcal{B}$. Hint: First show that $\mu(C) = \mu(\overline{C})$ for every open cube C and then appeal to the problem above.

Solution:We first claim that $\mu(R) > 0$ whenever R is a cube. By the assumption, $\exists A \in \mathcal{B}$ s.t.$\mu(A) > 0$. Moreover

$$A = \bigcup_{j=1}^{\infty} (A \cap B_j),$$

where $B_j = B_j(0)$ ball.

$$\mu(A) \le \sum_{j=1}^{\infty} \mu(A \cap B_j) \Rightarrow \exists A \cap B_j, \mu(A \cap B_j) > 0.$$

Therefore we may assume that $\exists$ bound set A s.t. $\mu(A) > 0$. By translational invariance, we may cover A with finitely many copy of R, then we know that $\mu(R) > 0$. By a problem in Exercise 3 we know that every open G can be written as

$$G = \bigcup_j R_j, R_j \text{ almost disjoint closed cubes }.$$

Again by the translational invariance of μ, we have the face of cube R are of

μ measure zero and

$$\mu(R) = \mu(\overline{R}).$$

Hence $\forall$ open G,

$$\mu(G) = \mathcal{L}^n(G)$$

By Problem 6, we are done.

(8) Let K be compact in $\mathbb{R}^n$ and $K^\varepsilon = \{x : \text{dist}(x, K) < \varepsilon\}$ be open. Show that $\mathcal{L}^n(K^\varepsilon) \to \mathcal{L}^n(K)$ as $\varepsilon \to 0$.

Solution: Since K is a bounded set due to compactness, K^ε is also bounded for any $\varepsilon > 0$. Observe that $K = \cap_k K^{1/k}$ also due to compactness of K. Since $\{K^{1/k}\}_{k=1}^\infty$ is a descending sequence of sets, one has

$$\mathcal{L}^n(K) = \mathcal{L}^n(\bigcap_{k=1}^\infty K^{1/k}) = \lim_{k\to\infty} \mathcal{L}^n(K^{1/k}).$$

(9) Let A and B be non-empty measurable sets in $\mathbb{R}^n$ such that $(1-\lambda)A + \lambda B$ is also measurable for all $\lambda \in (0, 1)$. Show that Brunn-Minkowski inequality is equivalent to either one of the following inequalities:

(a) $\mathcal{L}^n\big((1-\lambda)A + \lambda B\big) \geq (1-\lambda)\mathcal{L}^n(A) + \lambda\mathcal{L}^n(B)$.

(b) $\mathcal{L}^n\big((1-\lambda)A + \lambda B\big) \geq \min\big\{\mathcal{L}^n(A), \mathcal{L}^n(B)\big\}$.

Solution: By prop 3.2 of chapter 3, we know that for all Lebesgue measurable set A and and real number c, cA is also Lebesgue measurable and

$$\mathcal{L}^n(cA) = |c|^n \mathcal{L}^n(A).$$

Brunn-Minkowski inequality $\Rightarrow$ a):

$$\begin{aligned}\mathcal{L}^n\big((1-\lambda)A+\lambda B\big)^{1/n} &\geq \mathcal{L}^n((1-\lambda)A)^{1/n}+\mathcal{L}^n(\lambda B)^{1/n} \\ &= (1-\lambda)\mathcal{L}^n(A)^{1/n}+\lambda\mathcal{L}^n(B)^{1/n}.\end{aligned}$$

a) $\Rightarrow$ b):

$$\begin{aligned}\mathcal{L}^n\big((1-\lambda)A+\lambda B\big)^{1/n} &\geq (1-\lambda)\mathcal{L}^n(A)^{1/n}+\lambda\mathcal{L}^n(B)^{1/n} \\ &\geq (1-\lambda)\min\big\{\mathcal{L}^n(A),\mathcal{L}^n(B)\big\}+\lambda\min\big\{\mathcal{L}^n(A),\mathcal{L}^n(B)\big\} \\ &= \min\big\{\mathcal{L}^n(A),\mathcal{L}^n(B)\big\}.\end{aligned}$$

b) $\Rightarrow$ Brunn-Minkowski inequality: Now let A, B be measurable sets s.t. $A+B$ is also measureable. W.L.O.G., A and B are of finite measure. If A or B is measure zero, we are done. We may suppose

$$\mathcal{L}^n(A) \text{ and } \mathcal{L}^n(B)>0.$$

Let $J=\mathcal{L}^n(A)^{1/n}+\mathcal{L}^n(B)^{1/n}>0$, $\lambda=\dfrac{\mathcal{L}^n(B)^{1/n}}{J}\in(0,1)$, by inner regularity for sufficiently small ε, $\exists K_1\subseteq A, K_2\subseteq B$ non empty compact sets s.t.

$$\mathcal{L}^n(A\setminus K_1)<\varepsilon.$$

and

$$\mathcal{L}^n(B\setminus K_2)<\varepsilon.$$

$$
\begin{aligned}
\mathcal{L}^n\big(\frac{A+B}{J}\big) &\geq \mathcal{L}^n\big(\frac{K_1+K_2}{J}\big) \\
&= \mathcal{L}^n\big((1-\lambda)\frac{K_1}{J(1-\lambda)} + \lambda\frac{K_2}{J\lambda}\big) \\
&\geq \min\big\{\mathcal{L}^n\big(\frac{K_1}{J(1-\lambda)}\big), \mathcal{L}^n\big(\frac{K_2}{J\lambda}\big)\big\} \\
&\geq 1 - \frac{\varepsilon}{\min\big\{\mathcal{L}^n(A), \mathcal{L}^n(B)\big\}},
\end{aligned}
$$

and the result follows by taking $\varepsilon \to 0$.