# MATH5011 Real Analysis I

## Exercise 1 Suggested Solution

Notations in the notes are used.

(1) Show that every open set in $\mathbb{R}$ can be written as a countable union of mutually disjoint open intervals. Hint: First show that every point $x$ in this open set is contained in a largest open interval $I_x$. Next, for any $x, y$, $I_x$ and $I_y$ either coincide and disjoint. Finally, argue there are at most countably many such intervals.

Solution:

Let $V$ be open in $R$. Fix x $\in V$, $\exists$ at least one open interval $I$, x $\in I$, $I \subseteq V$. Let $I_\alpha = (a_\alpha, b_\alpha)$, $\alpha \in \mathcal{A}$, be all intervals with this property. Let

$$I_x = (a_x, b_x), a_x = \inf_\alpha a_\alpha, b_x = \sup_\alpha b_\alpha.$$

satisfy x $\in I_x$, $I_x \subseteq V$. It is obvious that $I_x \cap I_y \neq \phi \Rightarrow I_x = I_y$. So

$$V = \bigcup_{x \in V} I_x.$$

As you can pick a rational number in each $I_x$ and $Q$ is countable,

$$V = \bigcup_{x_j \in V} I_{x_j}.$$

(2) Let $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous. Show that $\Psi(f, g)$ are measurable for any measurable functions $f$, $g$. This result contains Proposition 1.3 as a special case.

Solution:

Note that every open set $G \subseteq \mathbb{R}^2$ can be written as a countable union of set of the form $V_1 \times V_2$ where $V_1, V_2$ open in $\mathbb{R}$. (Think of $V_1 \times V_2 = (a,b) \times (c,d), a, b, c, d \in Q$).

Let $G \subseteq \mathbb{R}$ be open. Then $\Phi^{-1}(G)$ is open in $\mathbb{R}^2$, so

$$\Phi^{-1}(G) = \bigcup_n (V_n^1 \times V_n^2),$$

Then

$$h^{-1}(\Phi^{-1})(G) = \bigcup_n h^{-1}(V_n^1 \times V_n^2) = \bigcup_n f^{-1}(V_n^1) \cap g^{-1}(V_n^2)$$

is measurable since $f$ and $g$ are measurable. Hence $h = (f, g)$.

(3) Show that $f : X \to \overline{\mathbb{R}}$ is measurable if and only if $f^{-1}([a,b])$ is measurable for all $a, b \in \overline{\mathbb{R}}$.

Solution:

By def $f : X \to \overline{R}$ is measurable if $f^{-1}(G)$ is measurable. $\forall G$ open in $\overline{R}$. Every open set $G$ in $\overline{R}$ can be written as acountable union of $(a,b)$, $[-\infty, a)$, $(b, \infty]$, $a, b \in R$. So f$f$ is measurable iff $f^{-1}(a,b), f^{-1}[-\infty, a), f^{-1}(b, \infty]$ are measurable.

$\Rightarrow$) Use

$$f^{-1}(a,b) = \bigcap_n f^{-1}(a - \frac{1}{n}, b + \frac{1}{n})$$

$$f^{-1}[-\infty, a) = \bigcap_n f^{-1}[-\infty, a + \frac{1}{n})$$

$$f^{-1}(b, \infty] = \bigcap_n f^{-1}(b - \frac{1}{n}, \infty]$$

$\Leftarrow$) Use

$$f^{-1}(a,b) = \bigcup_n f^{-1}[a - \frac{1}{n}, b + \frac{1}{n}]$$

$$f^{-1}[-\infty, a) = \bigcap_n f^{-1}[-\infty, a - \frac{1}{n}]$$

$$f^{-1}(b, \infty] = \bigcap_n f^{-1}[b + \frac{1}{n}, \infty].$$

(4) Let $f$, $g$, $f_k, k \geq 1$, be measurable functions from $X$ to $\overline{\mathbb{R}}$.

(a) Show that $\{x : f(x) < g(x)\}$ and $\{x : f(x) = g(x)\}$ are measurable sets.

(b) Show that $\{x : \lim_{k\to\infty} f_k(x)$ exists and is finite$\}$ is measurable.

Solution:

(a) Suffice to show $\{x : F(x) > 0\}$ and $\{x : F(x) = 0\}$ are measurable. If $F$ is measurable, use

$$\{x : F(x) > 0\} = F^{-1}(0, \infty]$$

$$\{x : F(x) = 0\} = F^{-1}[0, \infty] \cap F^{-1}[-\infty, 0]$$

(b) Since $g(x) = \limsup_{k\to\infty} f_k(x)$ and $\liminf_{k\to\infty} f_k(x)$ are measurable.

$$\{x : \lim_{k\to\infty} f_k(x) \text{ exists }\} = \{x : \liminf_{k\to\infty} f_k(x) = \limsup_{k\to\infty} f_k(x)\}$$

On the other hand, the set$\{x : g(x) < +\infty\}$ is also measurable, so is their intersection.

(5) There are two conditions (i) and (ii) in the definition of a measure $\mu$ on $(X, \mathcal{M})$. Show that (i) can be replaced by the "nontriviality condition": There exists some $E \in \mathcal{M}$ with $\mu(E) < \infty$.

Solution:

If $\mu$ is a measure satisfying the nontriviality condition and (ii), let $A_1 = E$, $A_i = \phi$ for $i \geq 2$ in ii),

$$\infty > \mu(E) = \sum_{i=1}^{\infty} \mu(A_i) \geq \mu(A_1) + \mu(A_2) = \mu(E) + \mu(\phi)$$

so $0 \geq \mu(\phi) \geq 0$. We have $\mu$ is a measure satisfying (i) and (ii).
if $\mu$ is a measure satisfying (i) and (ii), taking $E = \phi$, we have the nontriviality condition.

(6) Let $\{A_k\}$ be measurable and $\sum_{k=1}^{\infty} \mu(A_k) < \infty$ and

$$A = \{x \in X : x \in A_k \text{ for infinitely many } k\}.$$

(a) Show that $A$ is measurable.

(b) Show that $\mu(A) = 0$.

This is Borel-Cantelli lemma, google for more.
Solution

(a) Note that

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} A_k.$$

This is clearly measurable.

(b) Since $\sum_{k=1}^{\infty} \mu(A_k) < \infty$, we have $\sum_{k=n}^{\infty} \mu(A_k) \to 0$ as $n \to \infty$. For any n$\in N$, we have

$$A \subset \bigcup_{k \geq n} A_k$$

and so

$$\mu(A) \leq \sum_{k=n}^{\infty} \mu(A_k)$$

.Taking $n \to \infty$, we have $\mu(A) = 0$.

(7) Let $T$ be a map from a measure space $(X, \mathcal{M}, \mu)$ onto a set $Y$. Let $\mathcal{N}$ be the set of all subsets $N$ of $Y$ satisfying $T^{-1}(N) \in \mathcal{M}$. Show that the triple $(Y, \mathcal{N}, \lambda)$ where $\lambda(N) = \mu(T^{-1}(N))$ is a measure space.

Solution:

First, we show $\mathcal{N}$ is a $\sigma$-algebra in $Y$. This follows from the relations

$$T^{-1}(Y) = X,$$
$$T^{-1}(Y \setminus A) = X \setminus T^{-1}(A),$$
$$T^{-1}(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} T^{-1}(A_n),$$

for any $A, A_n \in \mathcal{N}$. Let $\{A_n\}_{n=1}^{\infty}$ be a disjoint countable collection of $\mathcal{N}$, then $\{T^{-1}(A_n)\}_{n=1}^{\infty}$ is a disjoint countable collection of $\mathcal{M}$, and

$$\lambda(\bigcup_{n=1}^{\infty} A_n) = \mu(T^{-1}(\bigcup_{n=1}^{\infty} A_n)) = \mu(\bigcup_{n=1}^{\infty} T^{-1}(A_n)) = \sum_{n=1}^{\infty} \mu(T^{-1}(A_n)) = \sum_{n=1}^{\infty} \lambda(A_n).$$

This proves the countably additivity of $\lambda$, and hence the triple $(Y, \mathcal{N}, \lambda)$ where $\lambda(N) = \mu(T^{-1}(N))$ is a measure space.

(8) In Theorem 1.6 we approximate a non-negative measurable function $f$ by an increasing sequence of simple functions from below. Can we approximate $f$ by a decreasing sequence of simple functions from above? A necessary condition is that $f$ must be bounded in $X$, that is, $f(x) \leq M$, $\forall x \in X$ for some $M$. Under this condition, show that this is possible.

Solution:

$f(X) \subseteq [0, M]$, we can divide $[0, M]$ into subintervals

$$I_j^k = [\frac{jM}{2^k}, \frac{(j+1)M}{2^k}),$$

for j=0,1,2....$2^k-2$

$$I^k_{2^k-1}=[\frac{(2^k-1)M}{2^k},M],$$

define $\varphi_k(t)=\dfrac{(j+1)M}{2^k}$ if t $\in I_j$. As $I^{k+1}_{2j}\cup I^{k+1}_{2j+1}\subseteq I^k_j$, $\varphi_k(t)$ is a decreasing sequence(so is $\varphi_k(f(x))$) and $\varphi_k(f(x))$ is simple function satisfying the following inequality :

$$\varphi_k(f(x))\geq f(x)\geq \varphi_k(f(x))-\frac{M}{2^k}.$$

Hence $\varphi_k(f(x))$ is a deceasing sequence of simple functions which converges uniformly to f over X.

(9) A measure space is *complete* if every subset of a *null set*, that is, a set of measure zero, is measurable. This problem shows that every measure space can be extended to become a complete measure. It will be used later.

Let $(X,\mathcal{M},\mu)$ be a measure space. Let $\widetilde{\mathcal{M}}$ contain all sets $E$ such that there exist $A,B\in\mathcal{M}$, $A\subset E\subset B$, $\mu(B\backslash A)=0$. Show that $\widetilde{\mathcal{M}}$ is a $\sigma$-algebra containing $\mathcal{M}$ and if we set $\widetilde{\mu}(E)=\mu(A)$, then $(X,\widetilde{\mathcal{M}},\widetilde{\mu})$ is a complete measure space.

Solution:

We see that $\widetilde{\mathcal{M}}$ contains $\mathcal{M}$ by taking $E=A=B$ for any $E\in\mathcal{M}$. Suppose $E_i\in\widetilde{\mathcal{M}}$, $B_i\subseteq E_i\subseteq A_i$ where $B_i,A_i\in\mathcal{M}$ and $\mu(A_i\backslash B_i)=0$, then

$$\bigcap_{i=1}^{\infty}B_i\subseteq\bigcap_{i=1}^{\infty}E_i\subseteq\bigcap_{i=1}^{\infty}A_i$$

and

$$\mu(\bigcap_{i=1}^{\infty}A_i\backslash\bigcap_{i=1}^{\infty}B_i)\leq\mu(\bigcup_{i=1}^{\infty}A_i\backslash B_i)\leq\sum_{i=1}^{\infty}\mu(A_i\backslash B_i)=0.$$

We have $\bigcap_{i=1}^{\infty} E_i$ is in $\widetilde{\mathcal{M}}$. If $A \supseteq E \supseteq B$, then

$$X \backslash A \subseteq X \backslash E \subseteq X \backslash B$$

and

$$\mu((X \backslash B) \backslash (X \backslash A)) = \mu(A \backslash B).$$

Hence $X \backslash E$ is in $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}$ is a $\sigma$ algebra. We check that $\widetilde{\mu}$ is a measure on $\widetilde{M}$. Obviously $\widetilde{\mu}(\phi) = 0$. Let $E_i$ be mutually disjoint $\widetilde{\mu}$ measurable set, $\exists B_i, A_i$ $\mu$ measurable s.t

$$A_i \subseteq E_i \subseteq B_i$$

and

$$\mu(B_i \setminus A_i) = 0.$$

Using above argument, we have $\mu(\bigcup_{i=1}^{\infty} B_i \setminus \bigcup_{i=1}^{\infty} A_i) = 0$, And $A_i$ are mutually disjoint,

$$\widetilde{\mu}(\bigcup_{i=1}^{\infty} E_i) = \mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) = \sum_{i=1}^{\infty} \widetilde{\mu}(E_i).$$

So $\widetilde{\mu}$ is a measure on $\widetilde{M}$.

Finally, we check that $\widetilde{\mu}$ is a complete measure, let $E$ be a $\widetilde{\mu}$ measurable and null set, for all subset $C \subseteq E$, we have $\exists A, B \in \mathcal{M}$ s.t.$A \subseteq E \subseteq B$ and $\mu(A) = \mu(B) = 0$. Therefore

$$\phi \subseteq C \subseteq B$$

and

$$\mu(B) = 0.$$

We have $C \in \widetilde{\mathcal{M}}$.

(10) Here we review Riemann integral. Let $f$ be a bounded function defined on $[a,b], a,b \in \mathbb{R}$. Given any partition $P = \big\{a = x_0 < x_1 < \cdots < x_n = b\big\}$ on $[a,b]$ and tags $z_j \in [x_j, x_{j+1}]$, there corresponds a *Riemann sum* of $f$ given by $R(f,P,\mathbf{z}) = \sum_{j=0}^{n-1} f(z_j)(x_{j+1} - x_j)$. The function $f$ is called *Riemann integrable* with integral $L$ if for every $\varepsilon > 0$ there exists some $\delta$ such that

$$\big|R(f,P,\mathbf{z}) - L\big| < \varepsilon,$$

whenever $\|P\| < \delta$ and $\mathbf{z}$ is any tag on $P$. (Here $\|P\| = \max_{j=0}^{n-1} |x_{j+1} - x_j|$ is the length of the partition.) Show that

(a) For any partition $P$, define its *Darboux upper* and *lower sums* by

$$\overline{R}(f,P) = \sum_j \sup\big\{f(x) : x \in [x_j, x_{j+1}]\big\}(x_{j+1} - x_j),$$

and

$$\underline{R}(f,P) = \sum_j \inf\big\{f(x) : x \in [x_j, x_{j+1}]\big\}(x_{j+1} - x_j)$$

respectively. Show that for any sequence of partitions $\{P_n\}$ satisfying $\|P_n\| \to 0$ as $n \to \infty$, $\lim_{n\to\infty} \overline{R}(f,P_n)$ and $\lim_{n\to\infty} \underline{R}(f,P_n)$ exist.

(b) $\{P_n\}$ as above. Show that $f$ is Riemann integrable if and only if

$$\lim_{n\to\infty} \overline{R}(f,P_n) = \lim_{n\to\infty} \underline{R}(f,P_n) = L.$$

(c) A set $E$ in $[a,b]$ is called *of measure zero* if for every $\varepsilon > 0$, there exists a countable subintervals $J_n$ satisfying $\sum_n |J_n| < \varepsilon$ such that $E \subset \bigcup_n J_n$. Prove Lebsegue's theorem which asserts that $f$ is Riemann integrable if and only if the set consisting of all discontinuity points of $f$ is a set of measure zero. Google for help if necessary.

Solution:

(a) It suffices to show: For every $\varepsilon > 0$, there exists some $\delta$ such that

$$0 \le \overline{R}(f, P) - \overline{R}(f) < \varepsilon,$$

and

$$0 \le \underline{R}(f) - \underline{R}(f, P) < \varepsilon,$$

for any partition $P$, $\|P\| < \delta$, where

$$\overline{R}(f) = \inf_P \overline{R}(f, P),$$

and

$$\underline{R}(f) = \sup_P \underline{R}(f, P).$$

.

If it is true, then $\lim_{n\to\infty} \overline{R}(f, P_n)$ and $\lim_{n\to\infty} \underline{R}(f, P_n)$ exist and equal to $\overline{R}(f)$ and $\underline{R}(f)$ respectively.

Given $\varepsilon > 0$, there exists a partition $Q$ such that

$$\overline{R}(f) + \varepsilon/2 > \overline{R}(f, Q).$$

Let $m$ be the number of partition points of $Q$ (excluding the endpoints). Consider any partition $P$ and let $R$ be the partition by putting together $P$ and $Q$. Note that the number of subintervals in $P$ which contain some partition points of $Q$ in its interior must be less than or equal to $m$. Denote the indices of the collection of these subintervals in $P$ by $J$.

We have

$$0 \leq \overline{R}(f,P) - \overline{R}(f,R) \leq \sum_{j\in J} 2M\Delta x_j \leq 2M \times m||P||,$$

where $M = \sup_{[a,b]} |f|$, because the contributions of $\overline{R}(f,P)$ and $\overline{R}(f,Q)$ from the subintervals not in $J$ cancel out. Hence, by the fact that $R$ is a refinement of Q,

$$\overline{R}(f) + \varepsilon/2 > \overline{R}(f,Q) \geq \overline{R}(f,R) \geq \overline{R}(f,P) - 2Mm||P||,$$

i.e.,

$$0 \leq \overline{R}(f,P) - \overline{R}(f) < \varepsilon/2 + 2Mm||P||.$$

Now, we choose

$$\delta < \frac{\varepsilon}{1+4Mm},$$

Then for $P$, $\|P\| < \delta$,

$$0 \leq \overline{R}(f,P) - \overline{R}(f) < \varepsilon.$$

Similarly, one can prove the second inequality.

(b) With the result in part a, it suffices to prove the following result: Let $f$ be bounded on $[a,b]$. Then f is Riemann integrable on $[a,b]$ if and only if $\overline{R}(f) = \underline{R}(f)$. When this holds, $L = \overline{R}(f) = \underline{R}(f)$.

According to the definition of integrability, when $f$ is integrable, there exists some $L \in \mathbb{R}$ so that for any given $\varepsilon > 0$, there is a $\delta > 0$ such that for all partitions $P$ with $||P|| < \delta$,

$$|R(f,P,z) - L| < \varepsilon/2,$$

holds for any tags z. Let $(P_1, z_1)$ be another tagged partition. By the triangle inequality we have

$$|R(f,P,z)-R(f,P_1,z_1)| \leq |R(f,P,z)-L|+|R(f,P_1,z_1)-L| < \varepsilon/2+\varepsilon/2 = \varepsilon.$$

Since the tags are arbitrary, it implies

$$\overline{R}(f,P) - \underline{R}(f,P) \leq \varepsilon.$$

As a result,

$$0 \leq \overline{R}(f) - \underline{R}(f) \leq \overline{R}(f,P) - \underline{R}(f,P) \leq \varepsilon.$$

Note that the first inequality comes from the definition of the upper/lower Riemann integrals. Since $\varepsilon > 0$ is arbitrary, $\overline{R}(f) = \underline{R}(f)$.

Conversely, using $\overline{R}(f) = \underline{R}(f)$ in part a, we know that for $\varepsilon > 0$, there exists a $\delta$ such that

$$0 \leq \overline{R}(f,P) - \underline{R}(f,P) < \varepsilon,$$

for all partitions $P$, $\|P\| < \delta$. We have

$$\begin{aligned} R(f,P,z) - \underline{R}(f) &\leq \overline{R}(f,P) - \underline{R}(f) \\ &\leq \overline{R}(f,P) - \underline{R}(f,P) \\ &< \varepsilon, \end{aligned}$$

and similarly,

$$\overline{R}(f) - R(f,P,z) \leq \overline{R}(f,P) - \underline{R}(f,P) < \varepsilon.$$

As $\overline{R}(f) = \underline{R}(f)$, combining these two inequalities yields

$$|R(f, P, z) - \underline{R}(f)| < \varepsilon,$$

for all $P$, $\|P\| < \delta$, so $f$ is integrable, where $L = \underline{R}(f)$.

(c) For any bounded $f$ on $[a, b]$ and $x \in [a, b]$, its **oscillation** at $x$ is defined by

$$\begin{aligned}\omega(f, x) &= \inf_{\delta} \{(\sup f(y) - \inf f(y)) : y \in (x - \delta, x + \delta) \cap [a, b]\} \\ &= \lim_{\delta \to 0^+} \{(\sup f(y) - \inf f(y)) : y \in (x - \delta, x + \delta) \cap [a, b]\}.\end{aligned}$$

It is clear that $\omega(f, x) = 0$ if and only if $f$ is continuous at $x$. The set of discontinuity of $f$, $D$, can be written as $D = \bigcup_{k=1}^{\infty} O(k)$, where $O(k) = \{x \in [a, b] : \omega(f, x) \geq 1/k\}$. Suppose that $f$ is Riemann integrable on $[a, b]$. It suffices to show that each $O(k)$ is of measure zero. Given $\varepsilon > 0$, by Integrability of $f$, we can find a partition $P$ such that

$$\overline{R}(f, P) - \underline{R}(f, P) < \varepsilon/2k.$$

Let $J$ be the index set of those subintervals of $P$ which contains some elements of $O(k)$ in their interiors. Then

$$\begin{aligned}\frac{1}{k} \sum_{j \in J} |I_j| &\leq \sum_{j \in J} (\sup_{I_j} f - \inf_{I_j} f) \Delta x_j \\ &\leq \sum_{j=1}^{n} (\sup_{I_j} f - \inf_{I_j} f) \Delta x_j \\ &= \overline{R}(f, P) - \underline{R}(f, P) \\ &< \varepsilon/2k.\end{aligned}$$

Therefore

$$\sum_{j \in J} |I_j| < \varepsilon/2.$$

Now, the only possibility that an element of $O(k)$ is not contained by one of these $I_j$ is it being a partition point. Since there are finitely many partition points, say $N$, we can find some open intervals $I'_1, ..., I'_N$ containing these partition points which satisfy

$$\sum |I'_i| < \varepsilon/2.$$

So $\{I_j\}$ and $\{I'_i\}$ together form a covering of $O(k)$ and its total length is strictly less than $\varepsilon$. We conclude that $O(k)$ is of measure zero.

Conversely, given $\varepsilon > 0$, fix a large $k$ such that $\frac{1}{k} < \varepsilon$. Now the set $O(k)$ is of measure zero, we can find a sequence of open intervals $\{I_j\}$ satisfying

$$O(k) \subseteq \bigcup_{j=1}^{\infty} I_j,$$

$$\sum_{j=1}^{\infty} |I_{i_j}| < \varepsilon.$$

One can show that $O(k)$ is closed and bounded, hence it is compact. As a result, we can find $I_{i_1}, ..., I_{i_N}$ from $\{I_j\}$ so that

$$O(k) \subseteq I_{i_1} \cup ... \cup I_{i_N},$$

$$\sum_{j=1}^{N} |I_j| < \varepsilon.$$

Without loss of generality we may assume that these open intervals are mutually disjoint since, whenever two intervals have nonempty intersection, we can put them together to form a larger open interval. Observe that $[a, b] \setminus (I_{i_1} \cup \cdots \cup I_{i_N})$ is a finite disjoint union of closed bounded intervals, call them $V_i's$, $i \in A$. We will show that for each $i \in A$, one can find a partition on each $V_i = [v_{i-1}, v_i]$ such that the oscillation of $f$ on each subinterval in this partition is less than $1/k$.

Fix $i \in A$. For each $x \in V_i$, we have

$$\omega(f, x) < \frac{1}{k}.$$

By the definition of $\omega(f, x)$, one can find some $\delta_x > 0$ such that

$$\sup\{f(y) : y \in B(x, \delta_x) \cap [a, b]\} - \inf\{f(z) : z \in B(x, \delta_x) \cap [a, b]\} < \frac{1}{k},$$

where $B(y, \beta) = (y - \beta, y + \beta)$. Note that $V_i \subseteq \bigcup_{x \in V_i} B(x, \delta_x)$. Since $V_i$ is closed and bounded, it is compact. Hence, there exist $x_{l_1}, \ldots, x_{l_M} \in V_i$ such that $V_i \subseteq \bigcup_{j=1}^{M} B(x_{i_j}, \delta_{x_{l_j}})$. By replacing the left end point of $B(x_{i_j}, \delta_{x_{l_j}})$ with $v_{i-1}$ if $x_{l_j} - \delta_{x_{l_j}} < v_{i-1}$, and replacing the right end point of $B(x_{i_j}, \delta_{x_{l_j}})$ with $v_i$ if $x_{l_j} + \delta_{x_{l_j}} > v_i$, one can list out the endpoints of $\{B(x_{l_j}, \delta_{l_j})\}_{j=1}^{M}$ and use them to form a partition $S_i$ of $V_i$. It can be easily seen that each subinterval in $S_i$ is covered by some $B(x_{l_j}, \delta_{x_{l_j}})$, which implies that the oscillation of $f$ in each subinterval is less than $1/k$. So, $S_i$ is the partition that we want.

The partitions $S_i$'s and the endpoints of $I_{i_1}, ..., I_{i_N}$ form a partition $P$ of $[a, b]$. We have

$$\begin{aligned}
\overline{R}(f, P) - \underline{R}(f, P) &= \sum_{I_{i_j}} (M_j - m_j)\Delta x_j + \sum (M_j - m_j)\Delta x_j \\
&\leq 2M \sum_{j=1}^{N} |I_{i_j}| + \frac{1}{k} \sum \Delta x_j \\
&\leq 2M\varepsilon + \varepsilon(b - a) \\
&= [2M + (b - a)]\varepsilon,
\end{aligned}$$

where $M = \sup_{[a,b]} |f|$ and the second summation is over all subintervals in $V_i, i \in A$. Hence $f$ is integrable on $[a, b]$.