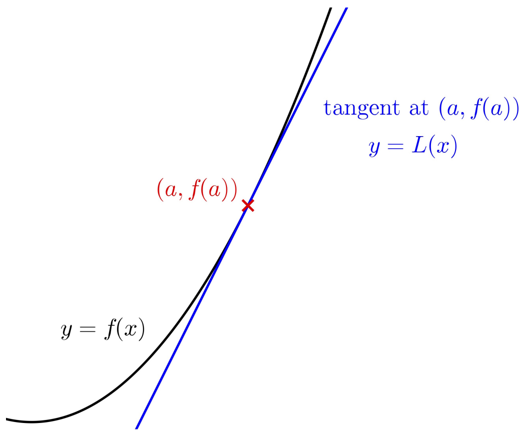
MATH 1510 Chapter 5

5.1 Linearization

Linearization of a function $f(x)$ at a point $(a, f(a))$ means finding the **linear approximation** of $f(x)$ at that point, i.e., finding the equation of the tangent at that point:



Apparently, $f(x) \approx L(x)$ if x is close to a.

Definition 5.1. If $f(x)$ is differentiable at a, then the **linearization**, or **linear approximation**, of $f(x)$ at a is:

$$y = L(x) = f(a) + f'(a)(x - a).$$

Example 5.2. Consider the function

$$f(x) = \arctan x.$$

To approximate $\arctan 0.1$, we consider the linearization of $f(x)$ at 0 :

$$L(x) = f(0) + f'(0)(x - 0) = x.$$

So,

$$\arctan 0.1 = f(0.1) \approx L(0.1) = 0.1.$$

(In fact, $\arctan 0.1 = 0.09966\ldots$)

Example 5.3. Approximate $\sqrt[3]{8.03}$ by linearizing a suitable function at an appropriately chosen point.

5.2 Mean Value Theorem

Theorem 5.4 (Rolles Theorem)**.** *Suppose $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then*

$$f'(c) = 0 \quad \textit{for some } c \in (a, b).$$

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Proof of Rolle's Theorem. It's trivial if $f(x)$ is a constant function. Assume $f(d) > f(a)$ for some $d \in (a, b)$. By Theorem 3.7 (Extreme Value Theorem (EVT)),

$$\exists c \in [a, b], M = f(c) = \max \{f(x) \mid x \in [a, b]\}$$

$$f(c) \geq f(d) > f(a) \implies c \in (a, b)$$

$$0 \geq \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = Rf'(c) = Lf'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0$$

Therefore, $f'(c) = 0$ as desired. The case when $f(d) < f(a)$ for some $d \in (a, b)$ can be handled by considering $-f(x)$. □

With Rolle's Theorem, we can derive the following important theorem:

Theorem 5.5 (Mean Value Theorem Lagrange)**.** *Suppose $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Then,*

$$f'(c) = \frac{f(b) - f(a)}{b - a} \textit{for some } c \in (a, b).$$

(It's a generalization of Rolle's Theorem)
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Proof of Mean Value Theorem (Lagrange). Let

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

Then, $g(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Moreover,

$$g(a) = f(a) = f(b) - \frac{f(b) - f(a)}{b - a}(b - a) = g(b)$$

By Theorem 5.4 (Rolle's Theorem),

$$\exists c \in (a, b), 0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$

and we are done. □

Example 5.6. Show that:

$$\sin x \leq x$$

for all $x \in [0, +\infty)$.

Let:

$$f(x) = x - \sin x.$$

It's then enough to show that $f(x) \geq 0$ for all $x \in [0, +\infty)$. First of all, $f(0) = 0$. So we may assume $x > 0$. Then, since $f(t)$ is continuous on $[0, x]$ and differentiable on $(0, x)$, we can apply Lagrange's MVT to obtain:

$$1 - \cos c = f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}.$$

Since $x > 0$, it's enough to show that $f'(c) = \dfrac{f(x)}{x} \geq 0$, which clearly holds because $1 - \cos c \geq 0$ for any c.

Exercise 5.7. Show that:

$$\arctan x \leq x$$

for all $x \in [0, +\infty)$.

Example 5.8. Show that:

$$x^2 + \ln(1 + x^2) \geq 2x \arctan x$$

for all $x \in [0, +\infty)$.

We may generalize Lagrange's MVT even further:

Theorem 5.9 (Cauchy's Mean Value Theorem)**.** *Suppose $f(x), g(x)$ are continuous on $[a, b]$ and differentiable on (a, b). If $g'(x) \neq 0$ on (a, b), then:*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)} \quad \textit{for some } c \in (a, b).$$

(It's a generalization of Lagrange's MVT by using $g(x) = x$)

Proof of Cauchy's Mean Value Theorem. (Sketch) Apply Theorem 5.4 (Rolle's Theorem) to

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$$

□

5.3 L'Hôpital's rule

Theorem 5.10 (LHôpitals rule)**.** *Suppose:*

1. *Around a point c (not necessarily at c), $f(x), g(x)$ are differentiable and $g'(x) \neq 0$,*
2. $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ *exists,*
3. $\lim_{x \to c} |f(x)|, \lim_{x \to c} |g(x)|$ *are both 0 or both $+\infty$ (DNE).*

Then,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.$$

- For condition 2, if $\lim_{x \to c} \frac{f'(x)}{g'(x)} = +\infty$ (DNE) instead, then we also have $\lim_{x \to c} \frac{f(x)}{g(x)} = +\infty$ (DNE). Same holds for $-\infty$.
- The theorem still holds if c is replaced by $+\infty$, $-\infty$ or one-sided limits: c^+ or c^-.

Proof of L'Hôpital's rule. (Sketch)

Case 1: $\lim_{x \to c} |f(x)| = \lim_{x \to c} |g(x)| = 0$.

Since $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$, we may define $f(c) = g(c) = 0$ so that $f(x), g(x)$ are continuous around c.

Let's consider the right hand limit: $x > c$. By applying Theorem 5.9 (Cauchy's Mean Value Theorem) on $f(x), g(x)$ on the interval $[c, x]$, we have:

$$\frac{f'(d)}{g'(d)} = \frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f(x)}{g(x)}$$

for some $d \in (c, x)$.

Therefore,

$$\lim_{x \to c^+} \frac{f(x)}{g(x)} = \lim_{x \to c^+} \frac{f'(d)}{g'(d)} = \lim_{x \to c^+} \frac{f'(x)}{g'(x)} = L$$

where $L = \lim_{x \to c} \frac{f'(x)}{g'(x)}$.

Similarly, we also have:

$$\lim_{x \to c^-} \frac{f(x)}{g(x)} = \lim_{x \to c^-} \frac{f'(x)}{g'(x)} = L.$$

Case 2: $\lim_{x \to c} |g(x)| = +\infty$

For any $x > c$, let:

$$m(x) = \inf_{\xi \in (c,x)} \frac{f'(\xi)}{g'(\xi)} \quad \text{and} \quad M(x) = \sup_{\xi \in (c,x)} \frac{f'(\xi)}{g'(\xi)}$$

Then,

$$\lim_{x \to c^+} m(x) = \lim_{x \to c^+} M(x) = L = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

On the other hand, for any $c < y < x$, by Theorem 5.9 (Cauchy's Mean Value Theorem), there exists $\xi \in (y, x)$ such that:

$$m(x) \le \frac{f'(\xi)}{g'(\xi)} = \frac{f(x) - f(y)}{g(x) - g(y)} \le M(x)$$

$$\implies m(x) \le \frac{\frac{f(x)}{g(y)} - \frac{f(y)}{g(y)}}{\frac{g(x)}{g(y)} - 1} \le M(x)$$

As $\lim_{y\to c^+} \dfrac{f(x)}{g(y)} = \lim_{y\to c^+} \dfrac{g(x)}{g(y)} = 0$, we have:

$$m(x) \le \liminf_{y\to c^+} \frac{f(y)}{g(y)} \le \limsup_{y\to c^+} \frac{f(y)}{g(y)} \le M(x)$$

Thus, by Theorem 2.12 (Squeeze Theorem),

$$\lim_{x\to c^+} m(x) = \liminf_{y\to c^+} \frac{f(y)}{g(y)} = \limsup_{y\to c^+} \frac{f(y)}{g(y)} = \lim_{x\to c^+} M(x) = L$$

Hence, $\lim_{y\to c^+} \dfrac{f(y)}{g(y)} = L$ and the result follows from an analogous argument on the left. □

In the computation of limits, it is clear that l'Hôpital's rule is useful for resolving indeterminate forms $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

For other indeterminate forms, we may need some manipulations first.

$0 \cdot \infty$ That is, $f(x)g(x)$ where $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = \pm\infty$ (DNE). In this case,

$$\begin{aligned} f(x)g(x) &= \frac{f(x)}{\frac{1}{g(x)}} \left(\frac{0}{0} \text{ form}\right) \\ &= \frac{g(x)}{\frac{1}{f(x)}} \left(\frac{\infty}{\infty} \text{ form}\right) \end{aligned}$$

(whichever works).

$0^0,\ (+\infty)^0$ **or** 1^∞ For the 0^0 case, that is, $f(x)^{g(x)}$ where $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$. If we apply "$\ln$ " on it, we would have

$$\ln\left(f(x)^{g(x)}\right) = g(x) \cdot \ln f(x) (0 \cdot \infty \text{ form})$$

We can then convert it into a fraction as above and apply l'Hôpital's rule. If

$$\lim_{x\to c} (g(x) \cdot \ln f(x)) = L,\ +\infty \text{ or } -\infty \text{ (DNE)},$$

then we can conclude that:

$$\lim_{x\to c} f(x)^{g(x)} = e^L,\ +\infty \text{ (DNE) or } 0 (e^{+\infty} = +\infty, e^{-\infty} = 0)$$

respectively.

The cases $(+\infty)^0$ and 1^∞ can be handled in a similar manner.

$(+\infty) - (+\infty)$

We will examine how to handle concrete cases of such forms in the examples below.

Example 5.11.

$$\lim_{x\to 0}\frac{x-\sin x}{x^3}$$

Example 5.12.

$$\lim_{x\to 0^+} x\ln x$$

Example 5.13.

$$\lim_{x\to -\infty}\left(\frac{x}{x+1}\right)^{(x^3)}$$

Example 5.14.

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$$

Solution. First, we rewrite $\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)$ as:

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{\cos^2 x}{\sin^2 x}\right) &= \lim_{x\to 0}\left(\frac{\sin^2 x - x^2\cos^2 x}{x^2\sin^2 x}\right)\\ &= \lim_{x\to 0}\left(\frac{\sin^2 x - x^2\cos^2 x}{x^4}\cdot\frac{x^2}{\sin^2 x}\right)\\ &= \lim_{x\to 0}\left(\frac{\sin^2 x(1+x^2) - x^2}{x^4}\cdot\frac{x^2}{\sin^2 x}\right)\end{aligned}$$

The limit $\lim_{x\to 0}\left(\frac{\sin^2 x(1+x^2)-x^2}{x^4}\right)$ corresponds to the indeterminate form $\frac{0}{0}$. We have:

$$\begin{aligned}\lim_{x\to 0}\left(\frac{\left(\sin^2 x(1+x^2)-x^2\right)'}{\left(x^4\right)'}\right) &= \lim_{x\to 0}\left(\frac{2\sin(x)\cos(x)(1+x^2)+2x\sin^2 x-2x}{4x^3}\right)\\ &= \lim_{x\to 0}\left(\frac{\sin(2x)(1+x^2)-2x+2x\sin^2 x}{4x^3}\right)\\ &= \lim_{x\to 0}\left(\frac{\sin(2x)(1+x^2)-2x}{4x^3}\right)+\underbrace{\lim_{x\to 0}\left(\frac{2x\sin^2 x}{4x^3}\right)}_{=\frac{1}{2}}\end{aligned}$$

The limit $\lim_{x\to 0}\left(\frac{\sin(2x)(1+x^2)-2x}{4x^3}\right)$ also corresponds to the indeterminate form $\frac{0}{0}$.

Differentiating numerator and denominator again, we have:

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{(\sin(2x)(1+x^2)-2x)'}{(4x^3)'}\right) &= \lim_{x\to 0}\left(\frac{2\cos(2x)(1+x^2)+2x\sin(2x)-2}{12x^2}\right)\\
&= \frac{1}{6}\lim_{x\to 0}\left(\frac{\cos(2x)-1}{x^2}\right)+\frac{1}{6}\lim_{x\to 0}\cos(2x)+\frac{1}{6}\lim_{x\to 0}\frac{\sin(2x)}{x}\\
&\text{(Observe that } \cos(2x)=1-2\sin^2(x))\\
&= \frac{1}{6}\lim_{x\to 0}\left(\frac{-2\sin^2 x}{x^2}\right)+\frac{1}{6}+\frac{1}{3}\\
&= -\frac{1}{3}\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2+\frac{1}{2}\\
&= -\frac{1}{3}+\frac{1}{2}=\frac{1}{6}.
\end{aligned}$$

Hence, by L'Hôpital's Rule, we have:

$$\lim_{x\to 0}\left(\frac{\sin^2 x(1+x^2)-x^2}{x^4}\right)=\frac{1}{6}+\frac{1}{2}=\frac{2}{3}$$

We conclude that:

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\cot^2 x\right)=\left(\lim_{x\to 0}\frac{\sin^2 x(1+x^2)-x^2}{x^4}\right)\cdot\underbrace{\left(\lim_{x\to 0}\frac{x}{\sin x}\right)^2}_{=1^2}=\frac{2}{3}$$

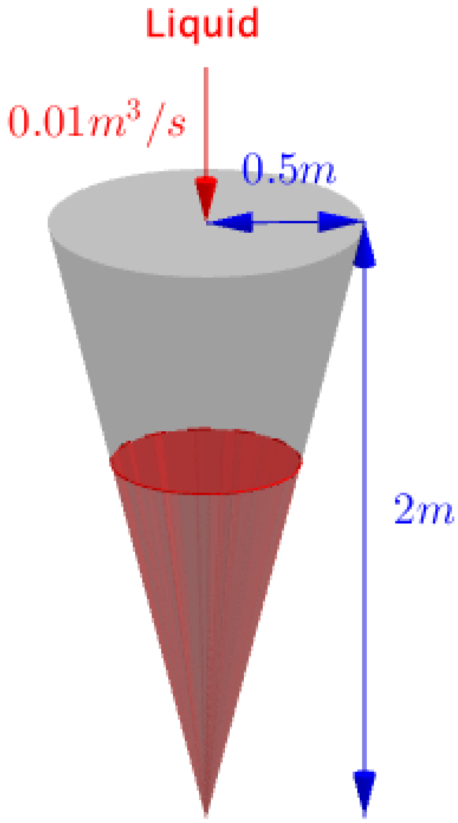
Example 5.15.

$$\lim_{x\to+\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

5.4 Rate of change

When some quantities change over time and they are related by equation(s), we can apply implicit differentiation to find how their rates of changes relate to each other.

Example 5.16. Suppose the height, radius of the base of a conical container are $2m$, $0.5m$ respectively and some liquid is poured into it at a speed of 0.01 (m^3/s).



Let $V(t), h(t), r(t)$ be the volume, height, radius of the base of the liquid inside the container at time t respectively. Then they are related by:

$$V(t) = \frac{1}{3}\pi r(t)^2 h(t)$$

Moreover, for $t > 0$,

$$\frac{r(t)}{h(t)} = \frac{0.5}{2} \implies 4r(t) = h(t) \implies V(t) = \frac{1}{48}\pi h(t)^3$$

Therefore, since the rate of change of volume of the liquid inside is $V'(t) = 0.01$,

$$0.01 = V'(t) = \frac{d}{dt}\left(\frac{1}{48}\pi h(t)^3\right) = \frac{1}{16}\pi h(t)^2 h'(t)$$

This shows that, for example, then the liquid reaches the height of $1.5m$, the rate at which liquid rises is:

$$h'(t) = \frac{16V'(t)}{\pi h(t)^2} = \frac{16 \cdot 0.01}{\pi \cdot (1.5)^2}\, m/s.$$

We can also see that at, for example, $t = 10$,

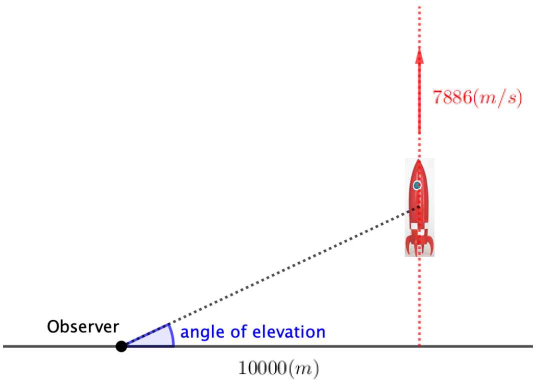
$$0.01(10) = V(10) = \frac{1}{48}\pi h(10)^3 \implies h(10) = \sqrt[3]{\frac{4.8}{\pi}} \approx 1.1518\,m$$

Hence,

$$0.01 = V'(10) = \frac{1}{16}\pi h(10)^2 h'(10) \implies h'(10) = \sqrt[3]{\frac{1}{5625\pi}} \approx 0.03839\,m/s$$

This shows that the liquid level will be rising at the rate of $0.03839\,m/s$ when $t = 10$.

Example 5.17. Suppose a rocket flies vertically upward at a speed of $7886(m/s)$ after launch. An observer is $10000(m)$ away from the launch pad.



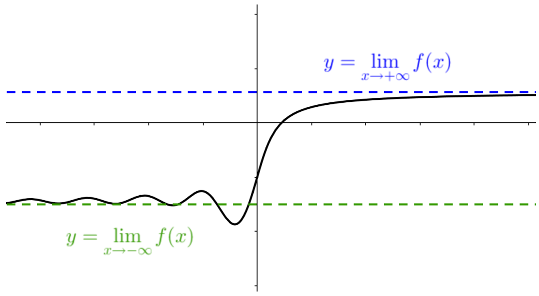
After 5 seconds, find

- the **angle of elevation** of the rocket (in degree, 2 decimal places)
- the rate of change of the angle of elevation (in degree/s, 2 decimal places)

5.5 Curve Sketching

Horizontal asymptote(s) of a function $f(x)$ (if any) can be found by computing

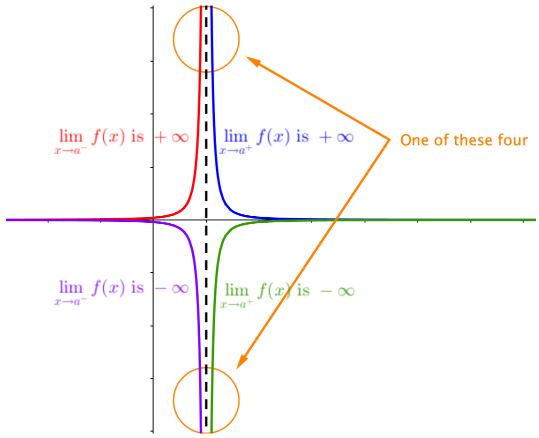
$$\lim_{x\to+\infty} f(x) \text{ and } \lim_{x\to-\infty} f(x)$$



Therefore, there are at most 2 horizontal asymptotes for any function. To verify that $x = a$ is a **vertical asymptote** of a function $f(x)$, we compute the one-sided limit:

Proposition 5.18.

$x = a$ *is a vertical asymptote of* $f(x) \iff \lim_{x\to a^+} f(x)$ *or* $\lim_{x\to a^-} f(x) = \pm\infty$ *(DNE)*



(Unlike horizontal asymptotes, there is no upper bound for how many vertical asymptotes a function has.)

Theorem 5.19 (Monotonicity Theorem)**.** *Suppose* $f(x)$ *is differentiable on an interval* I*. Then,*

$$\begin{aligned} f'(x) \geq 0 \text{ for all } x \in I &\iff f(x) \text{ is increasing on } I. \\ f'(x) \leq 0 \text{ for all } x \in I &\iff f(x) \text{ is decreasing on } I. \\ f'(x) > 0 \text{ for all } x \in I &\implies f(x) \text{ is strictly increasing on } I. \\ f'(x) < 0 \text{ for all } x \in I &\implies f(x) \text{ is strictly decreasing on } I. \end{aligned}$$

Proof of Monotonicity Theorem. We prove only the first case, the others can be proved similarly.

$\implies$ Suppose $f'(x) \geq 0$ for all $x \in I$. For any $x, y \in I$ such that $x < y$, since f is differentiable over $[x, y]$, by Lagrange's MVT,

$$\exists c \in (x, y), \frac{f(y) - f(x)}{y - x} = f'(c) \geq 0 \implies f(y) \geq f(x)$$

Thus, f is increasing on I.

$\impliedby$ Suppose f is increasing on I. Then, for any $x \in I$ which is not the right end-point,

$$f'(x) = Rf'(x) = \lim_{y \to x^+} \frac{f(y) - f(x)}{y - x} \geq 0$$

Similarly, for any $x \in I$ which is not the left end-point,

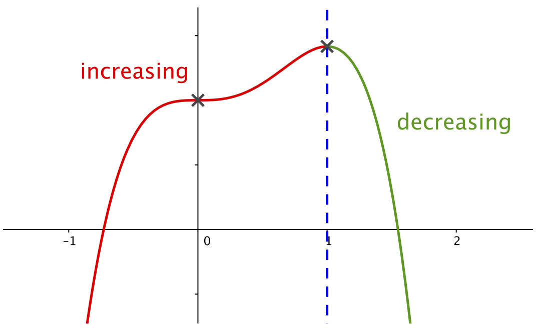
$$f'(x) = Lf'(x) = \lim_{y \to x^-} \frac{f(y) - f(x)}{y - x} \geq 0$$

Hence, $f'(x) \geq 0$ for all $x \in I$. The proof for decreasing function is similar. $\square$

Example 5.20. Given that:

x	$(-\infty, 0)$	0	$(0, 1)$	1	$(1, +\infty)$
$f'(x)$	$+$	0	$+$	0	$-$

Then, we know that $f(x)$ is increasing on $(-\infty, 1]$ and decreasing on $[1, +\infty)$. (Remember to include the end-points)



Definition 5.21. We say that a point $c \in D_f$ is a **local maximum** of a function $f(x)$ if

$$f(x) \leq f(c) \text{ around } c$$

and, similarly, we say that $c \in D_f$ is a **local minimum** of $f(x)$ if

$$f(x) \geq f(c) \text{ around } c.$$

We say that $c \in D_f$ is a **local extremum** if it's either a local maximum or local minimum.

local max. $f'(c) = 0$	IMAGE
local max. $f'(c)$ DNE	IMAGE
local min. $f'(c) = 0$	IMAGE
local min. $f'(c)$ DNE	IMAGE
Neither $f'(c) = 0$	IMAGE
Neither $f'(c)$ DNE	IMAGE

End-points can also correspond do local extrema: IMAGE

Definition 5.22. We say that $c \in D_f$ is a **critical point** of a function $f(x)$ if one of the following holds:

- $f'(c) = 0$
- $f(x)$ is not differentiable at c.

The above definition is motivated by the following important result:

Proposition 5.23. *If c is a local extremum of $f(x)$ over I, then:*

c is a critical point of $f(x)$ or an end-point of I.

Remark. In other words, any local extremum must either be a critical point or end-point. (Not vice versa. For example, 0 is a critical point of $f(x) = x^3$, but it's not a local extremum.)

Proof of Proposition 5.23. Suppose c is not an end-point of I and f is differentiable at c. Say, c is a local maximum of f. Then, $\exists \epsilon > 0$, c is a maximum of f over $(c - \epsilon, c + \epsilon)$. In this case,

$$Lf'(c) = \lim_{x \to c-} \frac{f(x) - f(c)}{x - c} \geq 0$$

$$Rf'(c) = \lim_{x \to c+} \frac{f(x) - f(c)}{x - c} \leq 0$$

Hence, $f'(c) = 0$ and c is a critical point. □

To classify whether a critical point is a local maximum, local minimum or neither, we may apply one of the following tests:

Theorem 5.24 (First Derivative Test)**.** *Let f be a function which is continuous on* $[a, b]$.

Suppose $c \in (a, b)$ is a critical point of f, and:

- *f is differentiable on $(a, c) \cup (c, b)$.*
- *If $f' \geq 0$ over (a, c) and $f' \leq 0$ over (c, b)*
  $\implies$ *c is a local maximum.*
- *If $f' \leq 0$ over (a, c) and $f' \geq 0$ over (c, b)*
  $\implies$ *c is a local minimum.*
- *Otherwise*
  $\implies$ *c is neither a local maximum nor minimum.*

Suppose $c = a$:

- *If $f' \leq 0$ over (c, b)*
  $\implies$ *c is a local maximum.*
- *If $f' \geq 0$ over (c, b)*
  $\implies$ *c is a local minimum.*

Suppose $c = b$:

- *If $f' \geq 0$ over (a, c)*
  $\implies$ *c is a local maximum.*

- *If $f' \leq 0$ over (a, c)*
  $\implies$ *c is a local minimum.*

Proof of First Derivative Test. See Theorem 6 in Appendix 4. □

Theorem 5.25 (Second Derivative Test)**.** *Suppose $f(x)$ is differentiable around c, twice-differentiable at c and $f'(c) = 0$. Then,*

(a)		$f''(c) < 0$
	$\implies$	*c is a local maximum.*
(b)		$f''(c) > 0$
	$\implies$	*c is a local minimum.*

(Inconclusive when $f''(c) = 0$)

Proof of Second Derivative Test. See Theorem 7 in Appendix 4. □

1^{st} test	$+ \quad -$	$+ \quad -$	$- \quad +$	$- \quad +$
2^{nd} test	$f''(c) < 0$	Can't apply	$f''(c) > 0$	Can't apply
	local max. IMAGE	local max. IMAGE	local min. IMAGE	local min. IMAGE

Example 5.26. Suppose:

$$f(x) = x^6 - 12x^5 + 36x^4, \quad \text{with } D_f = (-\infty, 5]$$

Then, f is twice-differentiable over D_f :

$$f'(x) = 6x^3(x-4)(x-6) \text{ and } Lf'(5) = -750$$
$$f''(x) = 6x^2(5x^2 - 40x + 72) \text{ and } Lf''(5) = -450$$

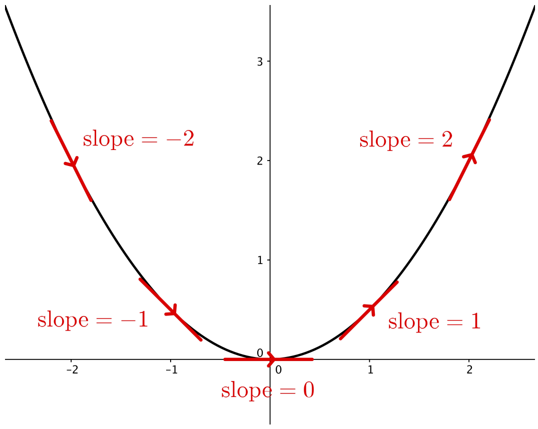
So the critical points are $0, 4$.

x	$(-\infty, 0)$	0	$(0, 4)$	4	$(4, 5)$	5
$f'(x)$	$-$	0	$+$	0	$-$	-750
1^{st} test		min.		max.		min.
$f''(x)$		0		$-768 < 0$		$-450 < 0$
2^{nd} test		inconclusive		max.		N.A.

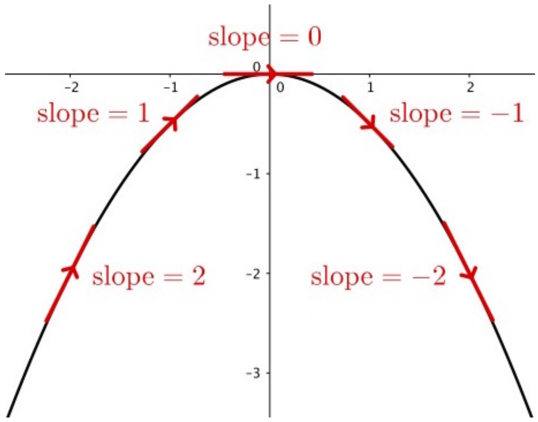
(To apply first derivative test at $x = 5$, computing $Lf'(5)$ is indeed unnecessary. On the other hand, second derivative test can't be applied at 5 because $Lf'(5) \neq 0$)

Definition 5.27. Suppose $f(x)$ is differentiable on an interval I. We say that $f(x)$ is **concave up** on I if $f'(x)$ is increasing on I. Similarly, we say that $f(x)$ is **concave down** on I if $f'(x)$ is decreasing on I.

Concave Up (slopes $\nearrow$ bending left)



Concave Down (slopes $\searrow$ bending right)



By Theorem 5.19 (Monotonicity Theorem), we have the following:

Theorem 5.28 (Concavity Theorem)**.** *Suppose $f(x)$ is twice differentiable on an interval I. Then,*

$$f''(x) \geq 0 \text{ for all } x \in I \iff f(x) \text{ is concave up on } I.$$
$$f''(x) \leq 0 \text{ for all } x \in I \iff f(x) \text{ is concave down on } I.$$

Definition 5.29 (Inflection Point)**.** Suppose a function $f(x)$ is continuous at c, differentiable on the left and right of c. We say that c is a **point of inflection** of $f(x)$ if $f(x)$ changes concavity at c (from up to down or down to up).

Proposition 5.30. *If c is a point of inflection of $f(x)$, then one of the following statements holds:*

- $f''(c) = 0$,
- *$f(x)$ is not twice differentiable at c.*

Proof of Proposition 5.30. See Proposition 4 in Appendix 4. □

Example 5.31. Recall Example 5.26 ,

$$f(x) = x^6 - 12x^5 + 36x^4, \quad \text{with } D_f = (-\infty, 5]$$

Then, f is twice-differentiable over D_f :

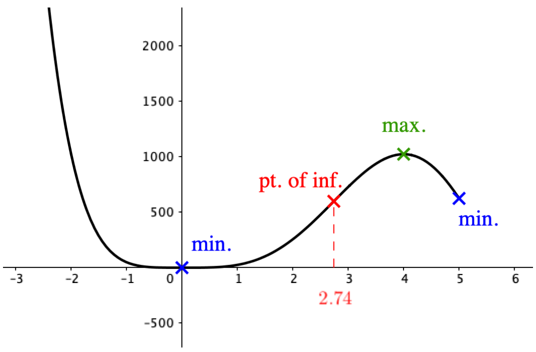
$$f'(x) = 6x^3(x-4)(x-6) \text{ and } Lf'(5) = -750$$
$$f''(x) = 6x^2(5x^2 - 40x + 72) \text{ and } Lf''(5) = -450$$

By solving $f''(x) = 0$, we know that the candidates for the point of inflection are

$$0,\ \alpha = \frac{2}{5}(10 - \sqrt{10}) \approx 2.74 \left(\frac{2}{5}(10 + \sqrt{10}) \approx 5.26 \text{ rejected}\right)$$

x	$(-\infty, 0)$	0	$(0, \alpha)$	α	$(\alpha, 5)$	5
$f''(x)$	$+$	0	$+$	0	$-$	-450
		no		pt. of inf.		

($x = 5$ is not a point of inflection because it's an end-point)



In general, to sketch the graph of a function, we consider the following:

- Domain
- x, y -intercepts
- Symmetry? (even, odd or neither)
- Asymptotes? (vertical, horizontal)
- Signs for $f'(x)$ (critical point, $\nearrow$, $\searrow$, local maximum, local minimum)
- Signs for $f''(x)$ (point of inflection, concavity)

Example 5.32. Sketch the graph of:

$$f(x) = \frac{x^2 + 5x + 4}{x^2}.$$

Example 5.33. Sketch the graph of:

$$f(x) = x^3 + x^{\frac{2}{3}}.$$

5.6 Optimization

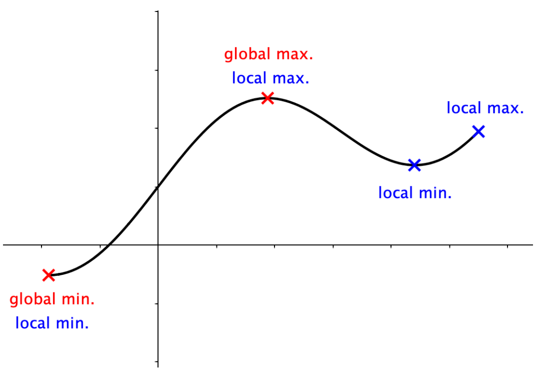
Definition 5.34. We say that $c \in D_f$ is a **global (absolute) maximum** of a function $f(x)$ on an interval I if

$$f(x) \leq f(c) \quad \text{for all } x \in I$$

and, similarly, we say that $c \in D_f$ is a **global (absolute) minimum** of a function $f(x)$ on an interval I if

$$f(x) \geq f(c) \quad \text{for all } x \in I$$

We say that $c \in D_f$ is a **global (absolute) extremum** if it's either a global maximum or global minimum.



From what we learned in the Chapter 3: Continuity, we know that:

Proposition 5.35.

- *If $f(x)$ is continuous on an interval $[a, b]$, then both global maximum and minimum exist.*
- *If $f(x)$ is a function on an interval I, then:*

$$\begin{aligned} & f(c) \text{ is a global max. (min.)} \\ \Longrightarrow\ & f(c) \text{ is a local max. (min.)} \\ \Longrightarrow\ & c \text{ is a critical point or end-point.} \end{aligned}$$

In other words, to find the global maximum/minimum of a continuous function $f(x)$ over an interval $[a, b]$, all we need to do is to find the critical points, end-points and compare their values.

Example 5.36. To find the global maximum and minimum of:

$$f(x) = x(x^2 - 1)$$

over the interval $[-1, 2]$, we first find its critical points. Since

$$f'(x) = 3x^2 - 1,$$

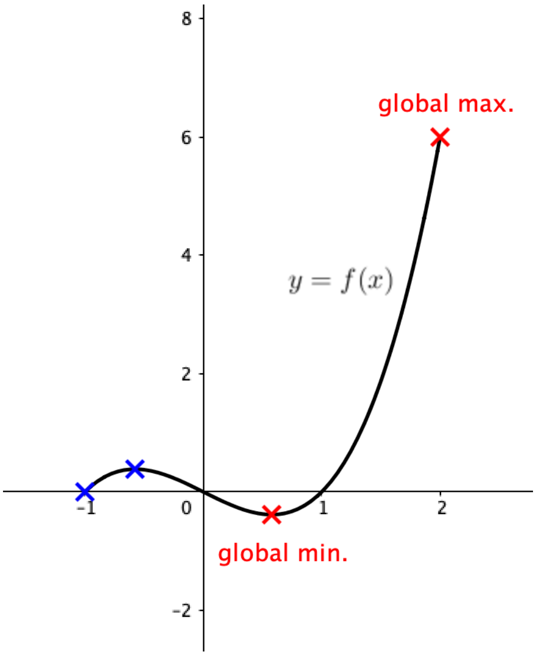
the critical points and end-points are

$$-1, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 2$$

with values:

$$f(-1) = 0,\ f\left(-\frac{1}{\sqrt{3}}\right) \approx 0.3849,\ f\left(\frac{1}{\sqrt{3}}\right) \approx -0.3849,\ f(2) = 6.$$

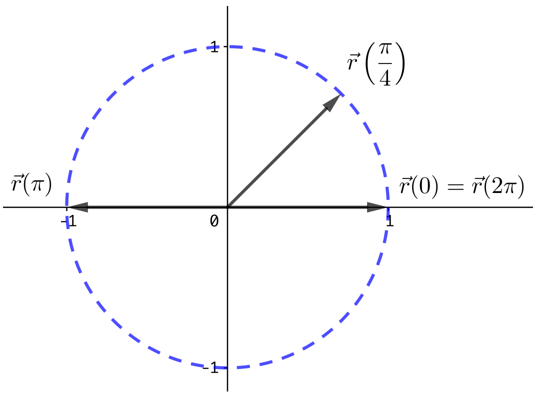
Hence, the global maximum value of $f(x)$ over $[-1, 2]$ is 6 (achieved by $x = 2$) and the global minimum value is $f\left(\frac{1}{\sqrt{3}}\right) \approx -0.3849$ (achieved by $x = \frac{1}{\sqrt{3}}$).



5.7 Derivatives of vector-valued functions

A **vector-valued function** is a function such that its outputs are vectors. For example,

$$\vec{r}(t) = (\cos t, \sin t) = (\cos t)\hat{\imath} + (\sin t)\hat{\jmath}$$



Definition 5.37. The derivative of a vector-valued function $\vec{r}(t)$ is defined as

$$\vec{r}\,'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

In practice, we simply differentiate the function component-wise:

Proposition 5.38. *For a vector-valued function:*

$$\vec{r}(t) = (x(t), y(t), z(t)) = x(t)\hat{\mathrm{i}} + y(t)\hat{\mathrm{j}} + z(t)\hat{\mathrm{k}},$$

if the functions $x(t), y(t), z(t)$ are differentiable at t, then:

$$\vec{r}\,'(t) = (x'(t), y'(t), z'(t)) = x'(t)\hat{\mathrm{i}} + y'(t)\hat{\mathrm{j}} + z'(t)\hat{\mathrm{k}}$$

Proof of Proposition 5.38. By definition,

$$\begin{aligned}
\vec{r}\,'(t) &= \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h} \\
&= \lim_{h\to 0} \frac{x(t+h)\hat{\mathrm{i}} + y(t+h)\hat{\mathrm{j}} + z(t+h)\hat{\mathrm{k}} - x(t)\hat{\mathrm{i}} - y(t)\hat{\mathrm{j}} - z(t)\hat{\mathrm{k}}}{h} \\
&= \left(\lim_{h\to 0} \frac{x(t+h) - x(t)}{h}\right)\hat{\mathrm{i}} + \left(\lim_{h\to 0} \frac{y(t+h) - y(t)}{h}\right)\hat{\mathrm{j}} + \left(\lim_{h\to 0} \frac{z(t+h) - z(t)}{h}\right)\hat{\mathrm{k}} \\
&= x'(t)\hat{\mathrm{i}} + y'(t)\hat{\mathrm{j}} + z'(t)\hat{\mathrm{k}}
\end{aligned}$$

□

Suppose the **displacement** (position relative to a fixed point) of an object at time t is given by $\vec{r}(t) = x(t)\hat{\mathrm{i}} + y(t)\hat{\mathrm{j}}$. Then, to find its velocity at time t_0, we start by considering

$$\frac{\text{displacement in 1 second}}{1} = \frac{\vec{r}(t_0+1) - \vec{r}(t_0)}{1}$$

This is the **average velocity** of the object over the time interval $[t_0, t_0+1]$.

Similarly,

$$\frac{\text{displacement in 0.5 second}}{0.5} = \frac{\vec{r}(t_0+0.5) - \vec{r}(t_0)}{0.5},$$

$$\frac{\text{displacement in 0.1 second}}{0.1} = \frac{\vec{r}(t_0+0.1) - \vec{r}(t_0)}{0.1}, \ldots$$

will give better and better approximations for the (*instantaneous*) velocity of the object at time t_0.

So, the object's **velocity** at time t_0 is equal to:

$$\lim_{h\to 0} \frac{\vec{r}(t_0+h) - \vec{r}(t_0)}{h} = \vec{r}\,'(t_0),$$

which captures both its speed and direction of motion at time t_0.

The **speed** of the object at time t_0 is simply the magnitude (or length) $|\vec{r}'(t_0)|$ of the velocity vector $\vec{r}'(t_0)$.

Similarly, the object's **acceleration** at time t_0 is equal to:

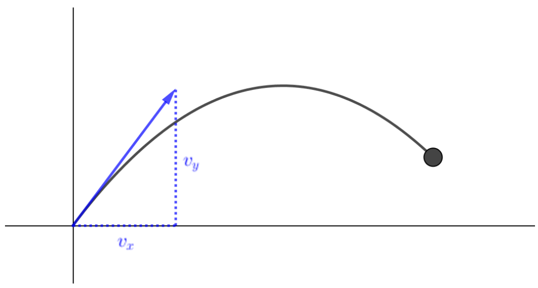
$$\lim_{h\to 0} \frac{\vec{r}'(t_0+h)-\vec{r}'(t_0)}{h} = \vec{r}''(t_0),$$

which captures both its change of speed and change of direction of motion at time t_0.

Example 5.39. According to **Newton's equations of motion** , the displacement of an object thrown at the ground level is given by

$$\vec{r}(t) = (v_x t)\hat{\imath} + \left(v_y t + \frac{1}{2}at^2\right)\hat{\jmath}$$

where t (in s) is the time elapsed, v_x (in m/s) is the initial rightward speed, v_y (in m/s) is the initial upward speed and $a = -9.8$ (in m/s^2) is the acceleration due to gravity. (This model takes $g = 9.8\,\mathrm{m/s^2}$ and neglects air resistance.)



Express the speed and magnitude of acceleration of the object at time t in terms of v_x, v_y, a.