MATH 1510 Chapter 4

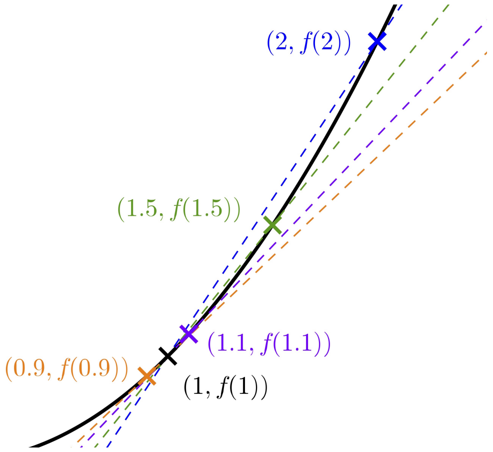
4.1 First principle

Consider the graph of the function $f(x) = x^2$ What is the slope of the **tangent** at the point $(1, 1)$?

A good starting point would be to approximate it by **secant lines**:

Secant line with	Slope
$(2, f(2))$	$\dfrac{f(2) - f(1)}{2 - 1} = 3$
$(1.5, f(1.5))$	$\dfrac{f(1.5) - f(1)}{1.5 - 1} = 2.5$
$(1.1, f(1.1))$	$\dfrac{f(1.1) - f(1)}{1.1 - 1} = 2.1$
$(0.9, f(0.9))$	$\dfrac{f(0.9) - f(1)}{0.9 - 1} = 1.9$

Secant Lines



Hence, slope of the tangent of $y = f(x)$ at $(1, f(1))$ should be:

$$\lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^2 - 1}{h} = 2$$

(The secant lines in Figure 4.1 correspond to h with values $1,\ 0.5,\ 0.1,\ -0.1$.)

Definition 4.1. The **derivative** of a function $f(x)$ at a point $x = a$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

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Example 4.2. Find $f'(a)$ if $f(x) = x^2$.

4.2 Differentiability

We say that a function $f(x)$ is **differentiable** at a point $x = a$ if the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. If so, such limit is denoted by $f'(a)$ or $\left.\dfrac{dy}{dx}\right|_a$.

Like limit, we also have one-sided derivatives:

Definition 4.3.

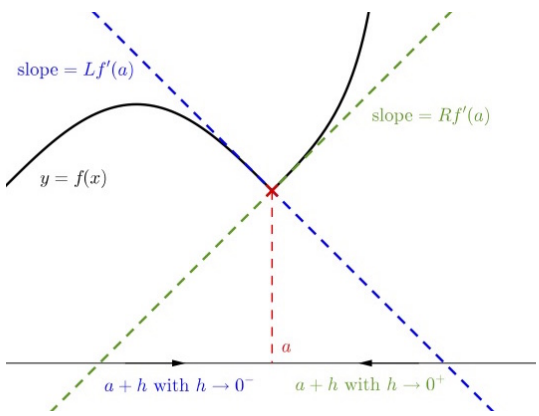
- **Left hand derivative**

$$Lf'(a) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

- **Right hand derivative**

$$Rf'(a) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

Geometrically, they may be viewed as the slopes of the tangents on the left and right, respectively:



Proposition 4.4. *A function f is differentiable at a if and only if $Lf'(a)$, $Rf'(a)$ both exist and are equal.*

If so, then:

$$f'(a) = Lf'(a) = Rf'(a) = \textit{slope of the tangent at } a.$$

Proof of Proposition 4.4. By definitions and the corresponding properties of one-sided limits. $\square$

Definition 4.5.

- We say that $f(x)$ is differentiable on (a, b) if $f(x)$ is differentiable at c for any $c \in (a, b)$.
- We say that $f(x)$ is differentiable on $[a, b)$ if $f(x)$ is differentiable on (a, b) and at a, in the sense that $Rf'(a)$ exists.
- We say that $f(x)$ is differentiable on $(a, b]$ if $f(x)$ is differentiable on (a, b) and at b, in the sense that $Lf'(b)$ exists.
- We say that $f(x)$ is differentiable on $[a, b]$ if $f(x)$ is differentiable on (a, b) and at both a, b.

Example 4.6. For the function:

$$f(x) = |x|,$$

we have:

$$\begin{aligned} Lf'(0) &= \lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1 \\ Rf'(0) &= \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h}{h} = 1 \end{aligned}$$

Therefore, the function is not differentiable at 0.

(One can show that $f(x)$ is differentiable on $(-\infty, 0) \cup (0, +\infty)$.)

Example 4.7. Is the function:

$$f(x) = \begin{cases} x^3 & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$$

differentiable at 0?

It's tempting to say that $Rf'(0) = 0$ for the function $f(x) = x^2$ because $f'(x) = 2x$. But in general we *cannot* assume that:

$$L'f(a) \neq \lim_{x \to a^-} f'(x) \quad \text{or} \quad R'f(a) \neq \lim_{x \to a^+} f'(x).$$

Consider the function:

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then, $Rf'(0) = 0$, but $\lim_{x\to 0^+} f'(x)$ DNE.

Differentiability is stronger than continuity:

Theorem 4.8. *If a function f is differentiable at a, and it is continuous at a.*

(The converse does not hold in general: $f(x) = |x|$ is continuous at 0, but not differentiable at 0)

Proof of Theorem 4.8. Since $g(x) = x - a$ is continuous over $\mathbb{R}$,

$$\begin{aligned} \lim_{x\to a}(f(x) - f(a)) &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a}(x - a) \\ &= f'(a)g(a) \\ &= 0 \\ \implies \lim_{x\to a} f(x) &= f(a). \end{aligned}$$

□

4.3 Derivative function and basic rules

By considering the slopes of the tangents at different points (assuming differentiability), we can consider the derivative of a function $f(x)$ as a function:

$$f' : x \mapsto f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

The domain of f' consists of those elements in the domain of f where f is differentiable.

We call $f'(x)$ the **derivative** of $f(x)$. It is also denoted by:

$$\frac{dy}{dx}, \quad \frac{d}{dx}f(x), \quad D_x f(x)$$

Example 4.9. Find $f'(x)$ if $f(x) = \sin x$.

Proposition 4.10.
- *If f, g are differentiable at a, then: $f \pm g$, $f \cdot g$ and $\dfrac{f}{g}$ (if $g(a) \neq 0$) are all differentiable at a.*

- *If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a*

(Some elementary functions are not differentiable at some points in their domains, e.g., the domain of $x^{\frac{1}{3}}$ is $\mathbb{R}$, but it's not differentiable at 0.)

Theorem 4.11. *For any differentiable functions f, g and constants $a, b \in \mathbb{R}$,*

- *(**Linearity**):*

$$(af(x) + bg(x))' = af'(x) + bg'(x)$$

- **Product Rule***:*

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

- **Quotient Rule***:*

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

  if $g(x) \neq 0$.

- **Chain Rule***:*

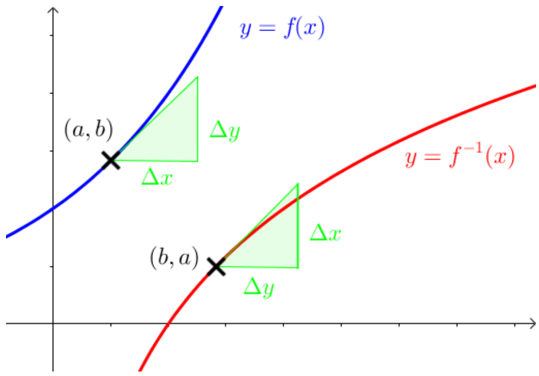
$$\frac{d}{dx}(g \circ f)(x) = g'(f(x)) \cdot f'(x)$$

Proof of Theorem 4.11. See Proposition 3 in Appendix 2. □

Theorem 4.12. *Suppose f^{-1} exists for a function f around a point a, $f(a) = b$ and f, f^{-1} are differentiable at a, b respectively. Then*

$$(f^{-1})'(b) = \frac{1}{f'(a)}$$

Proof of Theorem 4.12. See Theorem 8 in Appendix 2. $\square$



By the above rules, we can differentiate any complicated functions as long as we know the derivatives of the elementary functions.

4.4 Derivatives of elementary functions

Theorem 4.13 (Power Rule)**.** *For any constant $a \in \mathbb{R}$,*

$$\frac{d}{dx}(a) = 0, \frac{d}{dx}(x) = 1, \frac{d}{dx}(x^a) = ax^{a-1}$$

Proof of Power Rule. If a is a positive integer, then:

$$\begin{aligned}
\frac{d}{dx}x^a &= \lim_{h\to 0}\frac{(x+h)^a - x^a}{h} \\
&\quad (\text{Let } t = x+h) \\
&= \lim_{t\to x}\frac{t^a - x^a}{t-x} \\
&= \lim_{t\to x}\frac{(t-x)(t^{a-1} + t^{a-2}x + \cdots + tx^{a-2} + x^{a-1})}{t-x} \\
&= \lim_{t\to x}(t^{a-1} + t^{a-2}x + \cdots + tx^{a-2} + x^{a-1}) \\
&= ax^{a-1}
\end{aligned}$$

If a is a negative integer, then $x^a = \dfrac{1}{x^{-a}}$, and the theorem follows from an application of the qoutient rule.

If a is any real number, then for $x > 0$ we have:

$$x^a = e^{a \ln x}.$$

Hence:

$$\begin{aligned}
\frac{d}{dx}(x^a) &= \frac{d}{dx}(e^{a\ln x}) \\
&= e^{a\ln x} \cdot \frac{a}{x} \quad \text{(by the Chain Rule)} \\
&= x^a \cdot \frac{a}{x} \\
&= ax^{a-1}
\end{aligned}$$

(For derivatives of $e^x, \ln x$, see Propositions 4, 5 in Appendix 3) $\square$

Example 4.14. Find the derivative of:

- $$f(x) = \sqrt[3]{x} + \frac{1}{x}$$
- $$f(x) = \frac{x^2+1}{x+1}$$
- $$f(x) = \sqrt{x^2-1}$$

Theorem 4.15 (Derivatives of Trigonometric Functions)**.**

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \cos x & \frac{d}{dx}(\sec x) &= \sec x \tan x \\
\frac{d}{dx}(\cos x) &= -\sin x & \frac{d}{dx}(\csc x) &= -\csc x \cot x \\
\frac{d}{dx}(\tan x) &= \sec^2 x & \frac{d}{dx}(\cot x) &= -\csc^2 x
\end{aligned}$$

Proof of Derivatives of Trigonometric Functions. (Sketch) The fact that:

$$\frac{d}{dx}(\sin x) = \cos x$$

was handled in Example Example 4.9 . The derivative of $\cos x$ can be found by considering

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

The other four formulas can then be easily derived. $\square$

Theorem 4.16 (Derivatives of Inverse Trigonometric Functions)**.**

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\mathit{arcsec}\, x) = \frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\mathit{arccsc}\, x) = -\frac{1}{x\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\mathit{arccot}\, x) = -\frac{1}{1+x^2}$$

Proof of Derivatives of Inverse Trigonometric Functions.

$$\begin{aligned} y &= \arcsin x \\ \sin y &= x \\ \cos y &= \frac{dx}{dy} \\ \frac{dy}{dx} &= \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

Other formulas can be proved similarly. □

Theorem 4.17 (Derivatives of Exponential and Logarithmic Functions)**.**

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(\ln x) = \frac{1}{x}$$
$$\frac{d}{dx}(a^x) = (\ln a)a^x \qquad \frac{d}{dx}(\log_a x) = \frac{1}{(\ln a)x}$$

Proof of Derivatives of Exponential and Logarithmic Functions. (Sketch) For derivatives of $e^x, \ln x$, see Propositions 4, 5 in Appendix 3. The derivatives of a^x and $\log_a x$ can be derived easily from the facts that

$$a^x = e^{x \ln a} \text{and } \log_a x = \frac{\ln x}{\ln a}$$

□

Example 4.18. Find the derivative of:

- $$f(x) = \sec x \tan x$$

- $$f(x) = \arcsin(\cos x)$$

-
$$f(x) = \log_2(e^x + \sin x)$$

-
$$f(x) = \begin{cases} \ln x & \text{if } x \geq 1 \\ \cos\left(\frac{\pi x}{2}\right) & \text{if } 0 < x < 1 \\ 1 - x^2 & \text{if } x \leq 0 \end{cases}$$

4.5 Implicit differentiation

Consider the equation

$$x^2 + y^2 = 2.$$

How to find the slope of the tangent at the point $(1, 1)$?

Method 1

$$\begin{aligned} y &= \sqrt{2 - x^2} \quad \text{(upper half)} \\ y' &= -x(2 - x^2)^{-\frac{1}{2}} \\ y'(1) &= -1 \end{aligned}$$

So, the slope of the tangent is -1.

What if we can't solve for y?

Method 2

Consider y as a (differentiable) function of $x : y = y(x)$

$$\begin{aligned} x^2 + y(x)^2 &= 2 \\ \frac{d}{dx}(x^2 + y(x)^2) &= \frac{d}{dx}(2) \\ 2x + 2y(x)\frac{d}{dx}y(x) &= 0 \quad \text{(by the Chain rule)} \\ 2x + 2y(x)y'(x) &= 0 \end{aligned}$$

Therefore, $y' = -\dfrac{x}{y}$ and

$$y'(1) = -\frac{1}{1} = -1$$

This is what we called implicit differentiation.

Example 4.19. • Express y' in terms of x, y if:

$$y^3 + 7y = x^3$$

• Find $\left.\dfrac{dy}{dx}\right|_{(0,1)}$ if:

$$y \sin x = \ln y + x$$

4.6 Logarithmic differentiation

There is a trick called logarithmic differentiation that can sometimes simplify the process of differentiation.

Example 4.20. Find the derivative of

$$y = e^{5x} \sin 2x \cos x$$

Let's take "$\ln$" on both sides and use the properties of logarithm to simplify the expression:

$$\ln y = 5x + \ln(\sin 2x) + \ln(\cos x)$$

Then we differentiate both sides with respect to x :

$$\begin{aligned} \frac{d}{dx}(\ln y) &= \frac{d}{dx}(5x + \ln(\sin 2x) + \ln(\cos x)) \\ \frac{1}{y}y' &= 5 + \frac{2\cos 2x}{\sin 2x} + \frac{-\sin x}{\cos x} \end{aligned}$$

Hence,

$$y' = y(5 + 2\cot 2x - \tan x) = e^{5x} \sin 2x \cos x(5 + 2\cot 2x - \tan x)$$

Remark. One can also solve this problem by applying the product rule for three terms:

$$(f \cdot g \cdot h)' = f' \cdot g \cdot h + f \cdot g' \cdot h + f \cdot g \cdot h'$$

Example 4.21. Find the derivative of

$$y = x^x + \sin x$$

Applying "$\ln$ " directly will not help this time. So, instead, we handle the two terms on the right separately:

$$\begin{aligned}
y_1 &= x^x \\
\ln y_1 &= x \ln x \\
\frac{d}{dx}(\ln y_1) &= \frac{d}{dx}(x \ln x) \\
\frac{1}{y_1} y_1' &= \ln x + 1 \\
y_1' &= x^x(\ln x + 1)
\end{aligned}$$

$$y_2 = \sin x \implies y_2' = \cos x$$

Hence,

$$y' = y_1' + y_2' = x^x(\ln x + 1) + \cos x$$

Remark. One can also rewrite the expression as

$$x^x + \sin x = e^{x \ln x} + \sin x$$

and differentiate it directly.

Example 4.22. Find the derivative of:

- $$y = \sqrt{\frac{(x+1)(x+2)}{(x-1)(x-2)}}$$

- $$y = (\cos x)^{\sin x}$$

4.7 Higher Order Derivatives

We can differentiate a function more than once (assuming differentiability):

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = y'' = D_x^2 y$$

For any non-negative integer n,

$$\frac{d^n y}{dx^n} = y^{(n)} = D_x^n y$$

Remark. By convention, $\dfrac{d^0y}{dx^0} = y^{(0)} = y$

Example 4.23. Find $y^{(n)}$ if $y = \sin x$. Notice that

$$\begin{aligned} y^{(0)} &= \sin x \\ y^{(1)} &= \cos x \\ y^{(2)} &= -\sin x \\ y^{(3)} &= -\cos x \end{aligned}$$

and $y^{(4)} = \sin x = y^{(0)}$. That is, it repeats every four times. Therefore,

$$y^{(n)} = \begin{cases} \sin x & \text{if } n = 4m \\ \cos x & \text{if } n = 4m+1 \\ -\sin x & \text{if } n = 4m+2 \\ -\cos x & \text{if } n = 4m+3 \end{cases}$$

for any non-negative integer m.

Example 4.24. Find $\left.\dfrac{dy}{dx}\right|_{(1,0)}$ and $\left.\dfrac{d^2y}{dx^2}\right|_{(1,0)}$ if

$$y^3 + y = x^3 - x$$