MATH 1510 Chapter 3

3.1 Introduction

A highly imprecise "definition":

> A continuous function is one whose graph can be drawn without lifting your pencil from the paper.

Definition 3.1. A function $f(x)$ is **continuous** at $x = a$ if:

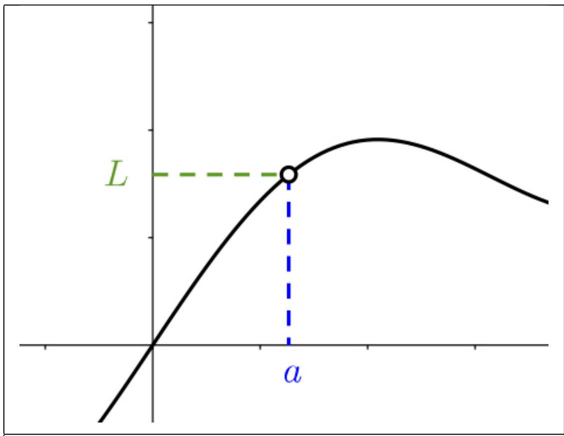
$$\lim_{x\to a} f(x) = f(a)$$

The equation "$\lim_{x\to a} f(x) = f(a)$ " actually means:

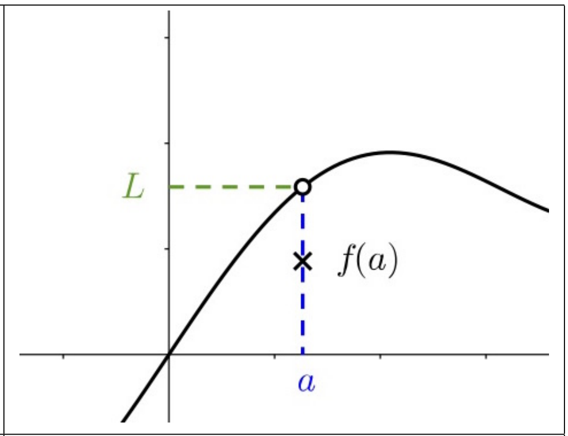
- $a \in D_f$
- $\lim_{x\to a} f(x)$ exists
- $\lim_{x\to a} f(x) = f(a)$

There are five possibilities:

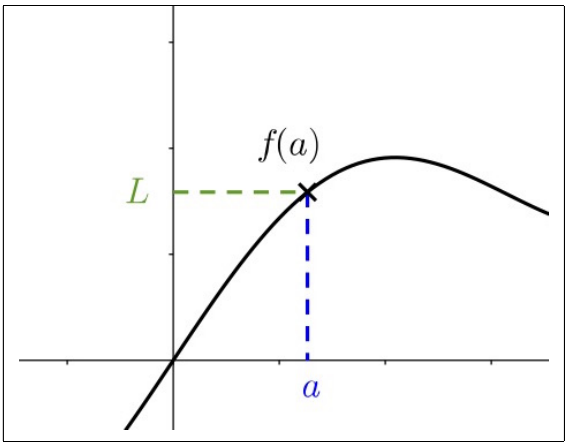
$$\lim_{x\to a} f(x) \begin{cases} = L & \begin{cases} a \notin D_f & \text{(case 1a)} \\ a \in D_f \text{ but } f(a) \neq L & \text{(case 1b)} \\ a \in D_f \text{ and } f(a) = L & \text{(case 1c)} \end{cases} \\ \\ \text{DNE} & \begin{cases} a \notin D_f & \text{(case 2a)} \\ a \in D_f & \text{(case 2b)} \end{cases} \end{cases}$$



(case 1a)

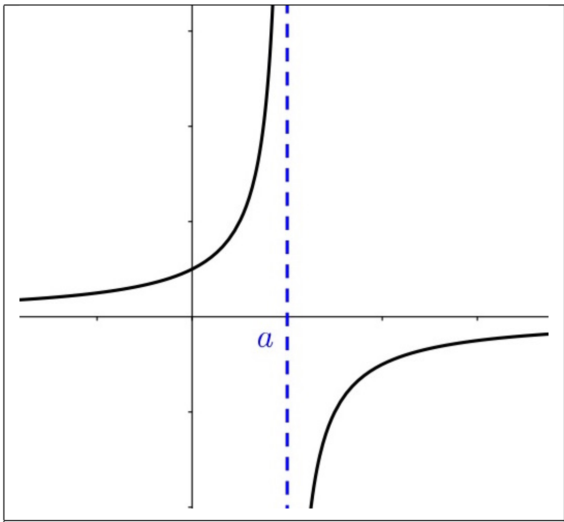


(case 1b)

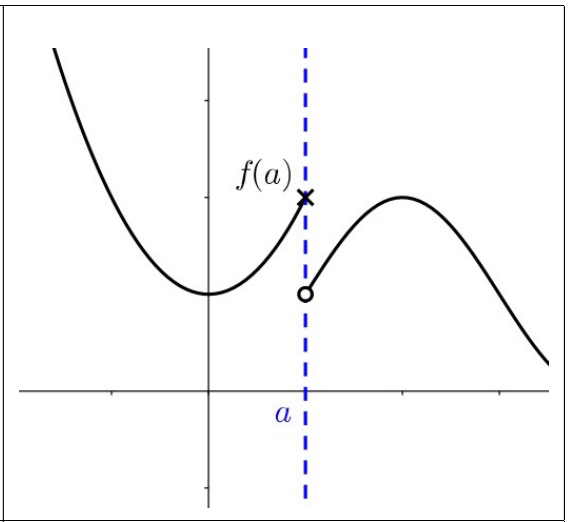


(case 1c)

That's why we define continuity this way



(case 2a)



(case 2b)

We can further define continuity over an interval:

Definition 3.2. We say that $f(x)$ is continuous on (a, b) if $f(x)$ is continuous at c for any $c \in (a, b)$. We say that $f(x)$ is continuous on $[a, b)$ if $f(x)$ is continuous on (a, b) and at a, in the sense that

$$\lim_{x \to a^+} f(x) = f(a)$$

We say that $f(x)$ is continuous on $(a, b]$ if $f(x)$ is continuous on (a, b) and at b, in the sense that

$$\lim_{x \to b^-} f(x) = f(b)$$

We say that $f(x)$ is continuous on $[a, b]$ if $f(x)$ is continuous on (a, b) and at both a, b.

Example 3.3. Given that:

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Show that $f(x)$ is continuous at 0.

Example 3.4. Given that:

$$f(x) = \begin{cases} xe^x & \text{if } x \geq 0 \\ \cos x + c & \text{if } x < 0 \end{cases}$$

Find the value(s) of the constant c such that f is continuous at 0.

3.2 Important results regarding continuity

Proposition 3.5.

- *If f, g are continuous at a, then:*

$$f \pm g, \quad f \cdot g, \quad \frac{f}{g} \text{ (if } g(a) \neq 0)$$

  are all continuous at a.

- *If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a*

- *"Elementary functions" (as in section 1.5) are all continuous on their domains.*

Proof of Proposition 3.5. See Proposition 1 in Appendix 2 and Theorem 3 in Appendix 4. □

Example 3.6. Consider:

$$\lim_{x \to 1}(x^2 + \sqrt{x} + 2^x)$$

Let:

$$f(x) = x^2 + \sqrt{x} + 2^x$$

Since the functions $x^2, \sqrt{x}, 2^x$ are elementary with implied domains $\mathbb{R}, [0, +\infty), \mathbb{R}$ respectively, f is continuous on $[0, +\infty)$, in particular, at $x = 1$. Hence,

$$\lim_{x \to 1}(x^2 + \sqrt{x} + 2^x) = \lim_{x \to 1} f(x) = f(1) = 4$$

That justifies why we could sometimes compute limits by direct substitution.

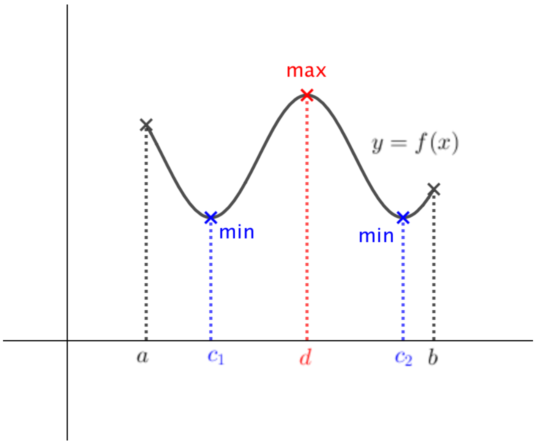
Theorem 3.7 (Extreme Value Theorem EVT)**.** *Suppose f is continuous on $[a, b]$. Then, f attains its maximum and minimum, i.e., there exist $c, d \in [a, b]$, such that:*

$$f(c) \leq f(x) \leq f(d)$$

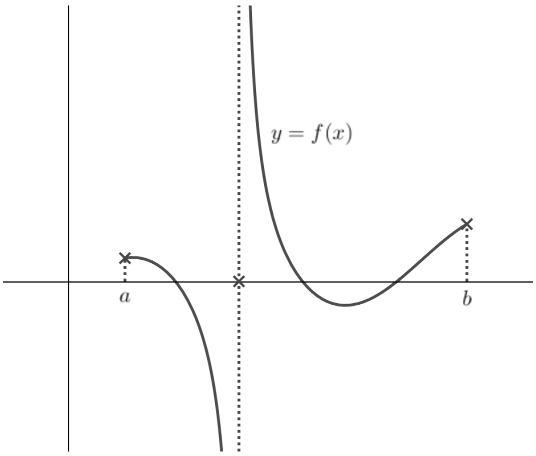
for all $x \in [a, b]$.

Proof of Extreme Value Theorem (EVT). See Theorem 2 in Appendix 2. □

Example:



Counterexample:

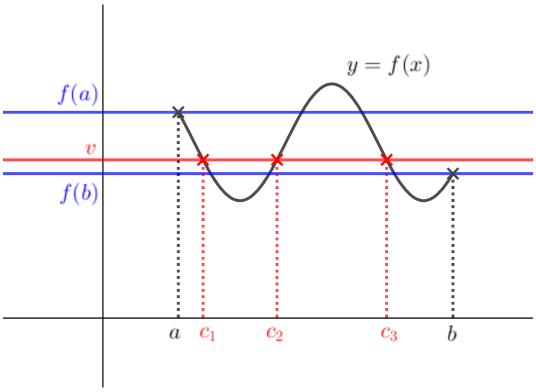


Theorem 3.8 (Intermediate Value Theorem IVT)**.** *If f is continuous on $[a, b]$, then, for any $v \in [f(a), f(b)]$ (or $[f(b), f(a)]$),*

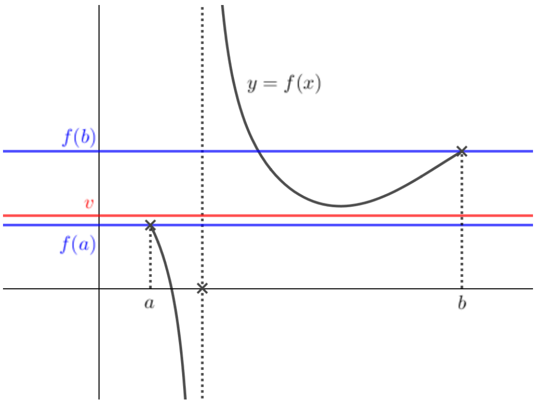
$$f(c) = v \quad \textit{for some } c \in [a, b].$$

Proof of Intermediate Value Theorem (IVT). See Theorem 3 in Appendix 2. □

Example:



Counterexample:



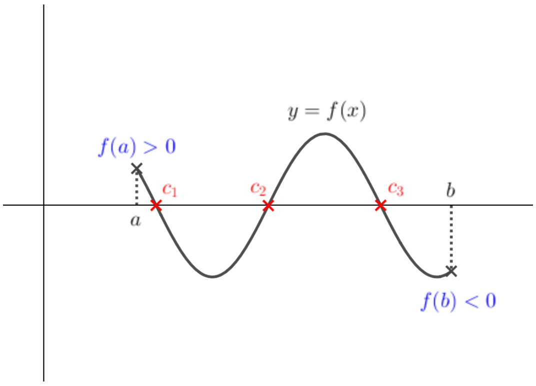
Theorem 3.9 (Bolzanos Theorem)**.** *Suppose f is continuous on $[a, b]$. If $f(a), f(b)$ have opposite signs, then:*

$$f(c) = 0 \quad \textit{for some } c \in (a, b).$$

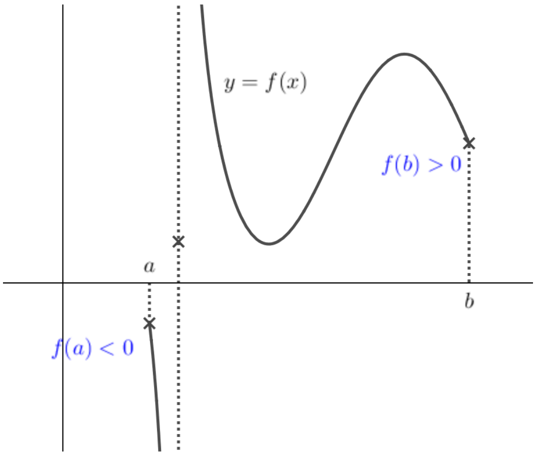
(This guarantees that there is at least one root in (a, b), but it is possible to have more.)

Proof of Bolzano's Theorem. Putting $v = 0$ in Theorem 3.8 (Intermediate Value Theorem (IVT)). □

Example:



Counterexample:



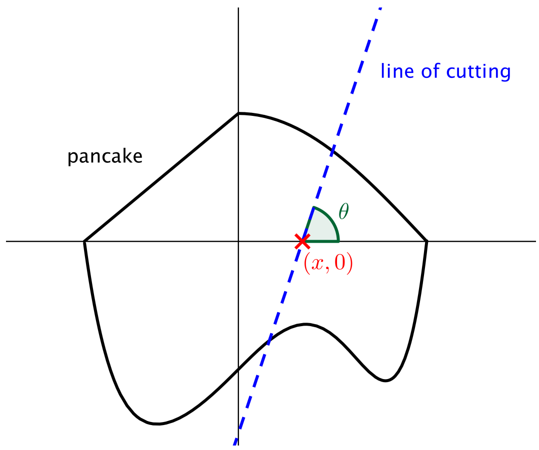
Remark. For Theorem Theorem 3.7 (Extreme Value Theorem (EVT)), Theorem 3.8 (Intermediate Value Theorem (IVT)) and Theorem 3.9 (Bolzano's Theorem), it is actually crucial that f is continuous on a closed interval $[a, b]$, instead of $(a, b]$, $[a, b)$ or (a, b).

Example 3.10. Show that

$$x^2 = \cos x$$

has a solution in $\mathbb{R}$.

Example 3.11 (Pancake Theorem)**.** Show that, no matter what shape a pancake is, what angle it's being cut at ($\theta \in (0, 180°)$), it can always be cut into two equal halves (by moving x).


line of cutting
pancake
θ
(x, 0)