Math 1510 Chapter 2

2.1 Limits of Functions on the Real Line

Informally, the **limit** $\lim_{x\to a} f(x)$ of a function f at a is a value which $f(x)$ approaches as x approaches a (but not at a).

Example 2.1. Consider $f(x) = \dfrac{x^2-4}{x-2}$. Note that the function f is not defined at 2.

For x near 2, we have:

x	$f(x)$
2.1	4.1
2.01	4.01
2.001	4.001
1.9	3.9
1.99	3.99
1.999	3.999

Observe that when x approaches 2 from either left of right, $f(x)$ appears to approach 4. Hence, the limit $\lim_{x\to a} f(x)$ should be 4.

This turns out to be true, and is not surprising, since we can rewrite $f(x)$ as follows:

$$f(x) = \begin{cases} \dfrac{(x+2)(x-2)}{x-2}, & \text{if } x \neq 2; \\ \text{undefined}, & \text{if } x = 2. \end{cases}$$
$$= \begin{cases} x+2, & \text{if } x \neq 2; \\ \text{undefined}, & \text{if } x = 2. \end{cases}$$

Hence, all along we have really been asking what $x+2$ tends to as x tends to 2.

2.1.1 Basic Properties

Theorem 2.2. *If* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist, then, for any constant* $k \in \mathbb{R}$*, we have:*

$$\lim_{x\to a}(f(x) \pm g(x)) = \left(\lim_{x\to a} f(x)\right) \pm \left(\lim_{x\to a} g(x)\right)$$
$$\lim_{x\to a}(f(x) \cdot g(x)) = \left(\lim_{x\to a} f(x)\right) \cdot \left(\lim_{x\to a} g(x)\right)$$
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad (\textit{If } \lim_{x\to a} g(x) \neq 0)$$
$$\lim_{x\to a} k f(x) = k\left(\lim_{x\to a} f(x)\right)$$
$$\lim_{x\to a} f(x)^k = \left(\lim_{x\to a} f(x)\right)^k \quad (\textit{If } \lim_{x\to a} f(x) > 0))$$

Example 2.3. Compute the following limits, if they exist:

- $\lim_{x\to -1} \dfrac{x^2-1}{x^2-5x-6}$

- $\lim_{x\to 4} \dfrac{2-\sqrt{x}}{16-x^2}$

2.2 One-Sided Limits

- We write $\lim_{x\to a^+} f(x) = L$ if $f(x)$ approaches L as x approaches a *from the right* . We call this L the **right limit** of f at a.

- Similarly, we write $\lim_{x\to a^-} f(x) = L$ if $f(x)$ approaches L as x approaches a *from the left* . We call this L the **left limit** of f at a.

The limit $\lim_{x\to a} f(x)$ is sometimes called the **double-sided limit** of f at a. It exists if and only if both one-sided limits exist and are equal to each other. In which case, we have:

$$\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x).$$

Example 2.4. Consider the function:

$$f(x) = \begin{cases} \sin x & \text{if } x > 0 \\ e^x & \text{if } x < 0 \\ -1 & \text{if } x = 0 \end{cases}$$

Its left-hand limit and right-hand limit at 0 are:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^x = e^0 = 1$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \sin x = \sin 0 = 0$$

Since the limits obtained by approaching from the left and right are different, $\lim_{x\to 0} f(x)$ DNE (does not exist).

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Exercise 2.5. Define

$$f(x) = \begin{cases} x - 1 & \text{if } 1 \le x \le 2, \\ 2x + 3 & \text{if } 2 < x \le 4, \\ x^2 & \text{otherwise.} \end{cases}$$

Compute $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$. Then, find $\lim_{x\to 2} f(x)$, if it exists.

Answers.

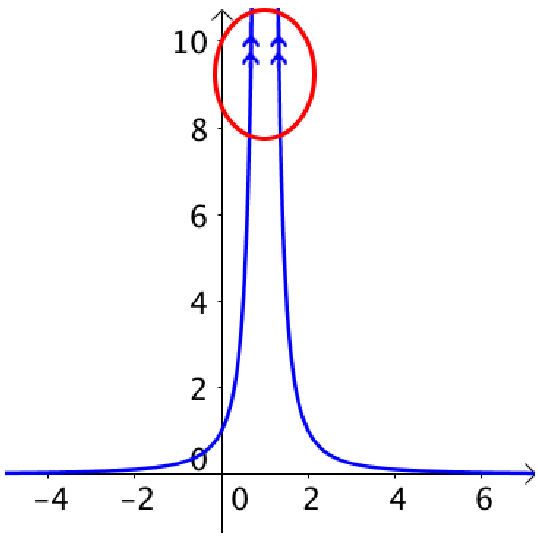
1.
$$\lim_{x\to 2^+} f(x) = 7$$
$$\lim_{x\to 2^-} f(x) = 1$$

2. Since $\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$, the double-sided limit $\lim_{x\to 2} f(x)$ does not exist.

2.2.1 Examples where the limit does not exist

Example 2.6.

- $\lim_{x \to 1} \frac{1}{(x-1)^2} = +\infty$ (DNE) .



- $\lim_{x \to 0} \frac{1}{x} =$ DNE, since $\lim_{x \to 0^-} \frac{1}{x} = -\infty$, while $\lim_{x \to 0^+} \frac{1}{x} = \infty$

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- $f(x) = \begin{cases} \sin x & \text{if } x > 0 \\ e^x & \text{if } x < 0 \\ -1 & \text{if } x = 0 \end{cases}$

  $\lim_{x \to 0} f(x)$ DNE.

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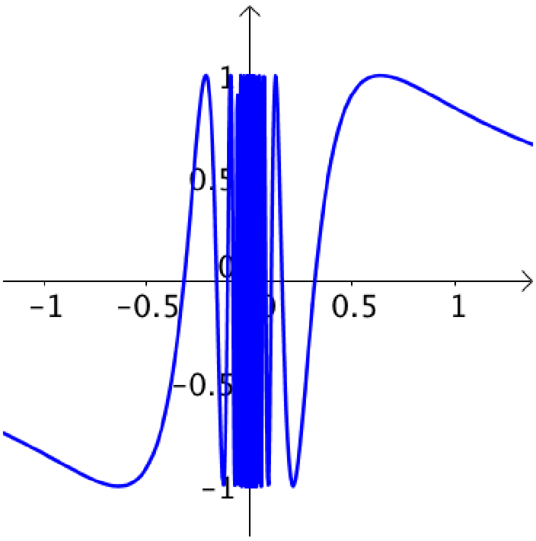
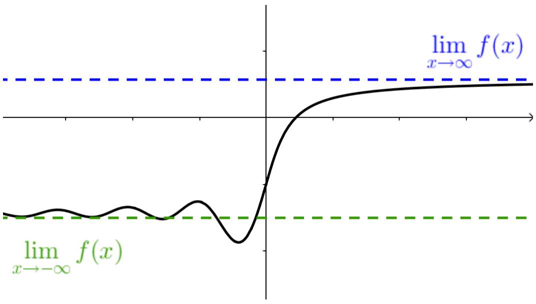
- $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ DNE.



2.3 Limit at $\pm\infty$

Informally, the limit $\lim_{x \to +\infty} f(x)$ of f at ∞ is the value, if it exists, that f approaches as x tends towards infinity.

Simlarly, the limit $\lim_{x \to -\infty} f(x)$ of f at $-\infty$ is the value, if it exists, that f approaches as x tends towards minus infinity.



Example 2.7.

- $$\lim_{x \to +\infty} \frac{2x + 1}{5x - 2}$$

- $$\lim_{x \to -\infty} \frac{-3x + 5}{9x^2 + 8x + 7}$$

- $$\lim_{x \to +\infty} \frac{x^2+1}{1-2x}$$

2.4 Sequences

A function f whose domain is $\mathbb{N}$ (all positive integers) is called a **sequence**. In this case we often use the notation:

$$a_n \quad \text{(instead of } f(n))$$

to denote the value of the function at $n \in \mathbb{N}$.

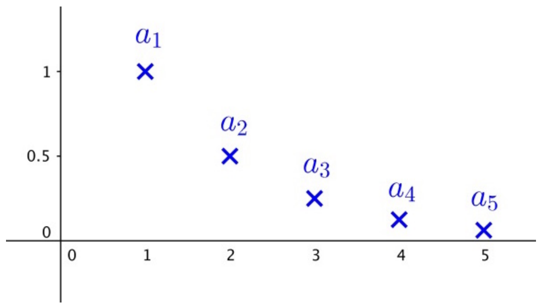
Example 2.8. Let:

$$a_n = \left(\frac{1}{2}\right)^{n-1},$$

then:

$$a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{4}, \ldots$$

We can graph such functions in the same way in which we graph functions who domains are the real line. The graph of a sequence looks like a collection of dots intead of curves:



And, like we do for functions on the real line, we could also consider the limit of a sequence at infinity (though it's not so meaningful to consider the limit of a sequence at a given point). In this example, it's clear that:

$$\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} \left(\frac{1}{2}\right)^{n-1} = 0.$$

Example 2.9.

$$\lim_{n\to+\infty} 2^n = +\infty \text{ (DNE)}$$
$$\lim_{n\to+\infty} \left(-\frac{1}{3}\right)^n = 0$$
$$\lim_{n\to+\infty} (-3)^n \quad \text{DNE}$$

Proposition 2.10. *Let a be a real number, then:*

$$\lim_{n\to\infty} a^n \quad \begin{cases} = +\infty (DNE) & \text{if } a > 1; \\ = 1 & \text{if } a = 1; \\ = 0 & \text{if } -1 < a < 1; \\ DNE & \text{if } a \leq -1. \end{cases}$$

Example 2.11. Find:

$$\lim_{n\to+\infty} \left(\frac{1}{3} + \frac{1}{9} + \cdots + \frac{1}{3^n}\right)$$

2.5 Squeeze Theorem for Functions on the Real Line

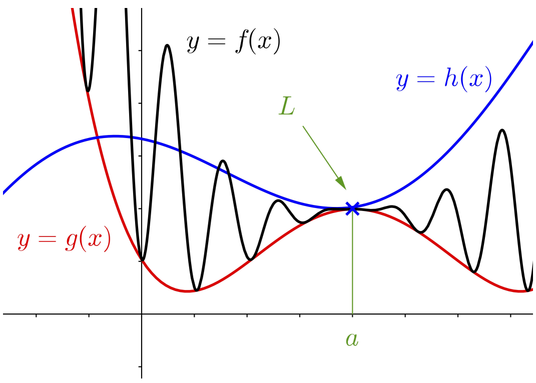
Theorem 2.12 (Squeeze Theorem)**.** *Let $a \in \mathbb{R}$, A an open neighborhood of a which does not necessarily contain a itself. Let $f, g, h : A \longrightarrow \mathbb{R}$ be functions such that:*

$$g(x) \leq f(x) \leq h(x) \quad \text{for all } x \in A,$$

and

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L.$$

Then, $\lim_{x\to a} f(x) = L$.



Similary,

Theorem 2.13. *If f, g, h are functions on $\mathbb{R}$ such that:*

$$g(x) \leq f(x) \leq h(x)$$

for all x sufficiently large, and

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = L,$$

then $\lim_{x\to\infty} f(x) = L$.

Exercise 2.14. Find the following limits, if they exist:

- $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$

- $\lim_{x\to\infty} \frac{\sin x}{x}$

- $\lim_{x\to\infty} \frac{x + \sin x}{x - \sin x}$

Theorem 2.15.

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Corollary 2.16.

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}\ .$$

Proof of Corollary 2.16.

$$\begin{aligned}
\lim_{x\to 0} \frac{1 - \cos x}{x^2} &= \lim_{x\to 0} \frac{1 - \cos x}{x^2} \cdot \left(\frac{1 + \cos x}{1 + \cos x}\right) \\
&= \lim_{x\to 0} \frac{1 - \cos^2 x}{x^2 (1 + \cos x)} \\
&= \lim_{x\to 0} \frac{\sin^2 x}{x^2 (1 + \cos x)} \\
&= \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2 \frac{1}{1 + \cos x} \\
&= 1^2 \cdot \frac{1}{1 + 1} = \frac{1}{2}
\end{aligned}$$

□

Corollary 2.17.

$$\lim_{x\to 0} \frac{1-\cos x}{x} = 0\ .$$

Exercise 2.18. Find the following limits, if they exist:

- $\lim_{x\to 0} \frac{\sin(5x)}{\tan(3x)}$

- $\lim_{x\to 0} \frac{x^3 \cos\left(\frac{1}{x}\right)}{\tan x}$

Definition 2.19. The constant e, known as **Euler's number**, is defined as the limit of a sequence:

$$e = \lim_{n\to +\infty} \left(1+\frac{1}{n}\right)^n$$

Theorem 2.20.

$$\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = \lim_{x\to -\infty} \left(1+\frac{1}{x}\right)^x = \lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$$

Corollary 2.21.

$$\lim_{x\to\infty} \left(1-\frac{1}{x}\right)^x = \lim_{x\to 0} (1-x)^{\frac{1}{x}} = \frac{1}{e}$$

For all $a \in \mathbb{R}$*,*

$$\lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x = e^a$$

Exercise 2.22. Find:

$$\lim_{x\to\infty} \left(\frac{x+1}{x-1}\right)^x$$

2.6 Indeterminate forms

While computing the limit (or one-sided limit) of $f(x)$ at $x = a$ (or $\pm\infty$), one might "substitute" $x = a$ (or $\pm\infty$) into $f(x)$ and get one of the followings:

$$\frac{0}{0},\ \frac{\pm\infty}{\pm\infty},\ 0\cdot(\pm\infty),\ \infty-\infty,\ \infty^0,\ 0^0,\ 1^{\pm\infty}$$

which are called **indeterminate forms** . In this case, we need to simplify/alter $f(x)$:

1.

$$\lim_{x \to 1} \frac{x^2 - 2x + 1}{x^2 - 3x + 2}$$

2.

$$\lim_{x \to 0} \frac{1 - \sqrt{x+1}}{x}$$

3.

$$\lim_{x \to -\infty} \frac{|x|}{x}$$

4.

$$\lim_{x \to +\infty} \frac{4x^3 - \sqrt{x^{10} - 8}}{(x+5)^2}$$

5.

$$\lim_{x \to +\infty} (x - \sqrt{x^2 + x})$$

6.

$$\lim_{x \to 0^+} \ln x$$

7.

$$\lim_{x \to -2} \frac{x^3 - 4x^2 - 7x + 10}{x + 2}$$

8.

$$\lim_{n \to +\infty} \frac{100^n}{n!}$$

9.

$$\lim_{x \to 0} \frac{\cos 2x - 1}{\sin^2 x}$$

10.

$$\lim_{x \to +\infty} \cos\left(\frac{\sin x}{x}\right)$$