THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1510 Calculus for Engineers
by Dr. Liu Chun Lung (Kelvin)

Appendix 4: Misc. Results

Proposition 1

$$\lim_{x\to\infty} a^x = \begin{cases} \infty \quad \text{(DNE)} & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } 0 < a < 1 \end{cases}, \quad \lim_{x\to\infty} x^a = \begin{cases} \infty \quad \text{(DNE)} & \text{if } a > 0 \\ 1 & \text{if } a = 0 \\ 0 & \text{if } a < 0 \end{cases}$$

where $x \in \mathbb{R}$.

Proof

$\lim_{x\to\infty} a^x$

The statement is trivial when $a = 1$.

When $a > 1$, we have $\ln a > 0$. Notice that

$$a^x = e^{x \ln a}$$

By continuity of e^x and $(\ln a)x$,

$$\lim_{x\to\infty} a^x = \lim_{x\to\infty} e^{x\ln a} = e^{\infty} = \infty$$

Similarly, when $1 > a > 0$, we have $\ln a < 0$.

$$\lim_{x\to\infty} a^x = \lim_{x\to\infty} e^{x\ln a} = e^{-\infty} = 0$$

$\lim_{x\to\infty} x^a$

The statement is trivial when $a = 0$. As above, when $a > 0$, by continuity of $e^x, \ln x$ and ax,

$$\lim_{x\to\infty} x^a = \lim_{x\to\infty} e^{a\ln x} = e^{\infty} = \infty$$

When $a < 0$,

$$\lim_{x\to\infty} x^a = \lim_{x\to\infty} e^{a\ln x} = e^{-\infty} = 0$$

□

Proposition 2

$$\lim_{n\to\infty} a^n \begin{cases} = \infty \quad \text{(DNE)} & \text{if } a > 1 \\ = 1 & \text{if } a = 1 \\ = 0 & \text{if } -1 < a < 1 \\ \text{DNE, neither } \pm\infty & \text{if } a \le -1 \end{cases}, \lim_{n\to\infty} n^a = \begin{cases} \infty \quad \text{(DNE)} & \text{if } a > 0 \\ 1 & \text{if } a = 0 \\ 0 & \text{if } a < 0 \end{cases}$$

where $n \in \mathbb{Z}^+$.

Proof

$\lim\limits_{n\to\infty} a^n$

Notice that $(n)_{n\in\mathbb{Z}^+}$ is a sequence with $\lim\limits_{n\to\infty} n = \infty$. By sequential criterion, the statements hold when $a > 0$. The statement is trivial when $a = 0$.

When $0 > a > -1$, the result follows by applying squeeze theorem on

$$-|a|^n \le a^n \le |a|^n$$

When $-1 \ge a$, observe that

$$|a^{2n} - a^{2n+1}| = |a|^{2n}|1-a| \ge 2$$

Assume $\lim\limits_{n\to\infty} a^n = L$. Then,

$$\exists N, \forall n > N, \quad |a^n - L| \le \frac{1}{2}$$

$$\implies 2 \le |a^{2n} - a^{2n+1}| = |a^{2n} - L + L - a^{2n+1}| \le |a^{2n} - L| + |L - a^{2n+1}| \le 1,$$

which is a contradiction. Thus, $\lim\limits_{n\to\infty} a^n$ DNE. Moreover, assume $\lim\limits_{n\to\infty} a^n = \infty$. Then,

$$\exists N, \forall n > N, \quad a^n > 1 \implies 1 < a^{2n+1} = |a|^{2n}a < 0,$$

which is a contradiction. A similar argument will show that $\lim\limits_{n\to\infty} a^n = -\infty$ is impossible too.

$\lim\limits_{n\to\infty} n^a$

As above, the results follow from sequential criterion. □

Theorem 1

Suppose $f(x)$ is a function such that $\lim\limits_{x\to a} f(x)$ exist. $\forall c \in \mathbb{R}$,

$$\lim_{x\to a} f(x)^c \quad = \quad \left(\lim_{x\to a} f(x)\right)^c \quad \text{provided that } \lim_{x\to a} f(x) > 0$$

The same hold if a is replaced by a^+, a^-, ∞ or $-\infty$.

Proof

Suppose $\lim\limits_{x\to a} f(x) = L > 0$.

$$\exists \delta > 0, \forall x \text{ such that } 0 < |x-a| < \delta, \quad |f(x) - L| < \frac{L}{2} \implies f(x) > \frac{L}{2} > 0$$

So $f(x)^c$ is well-defined for x around a.

Given sequence $(a_n)_{n\in\mathbb{Z}^+}$ such that $a_n \ne a$ and $\lim\limits_{n\to\infty} a_n = a$, by sequential criterion and continuity of $\ln x$ at L,

$$\lim_{n\to\infty} f(a_n) = L \implies \lim_{n\to\infty} (c\ln f(a_n)) = c\ln L$$

By continuity of e^x at $c\ln L$,

$$\lim_{n\to\infty} f(a_n)^c = \lim_{n\to\infty} e^{c\ln f(a_n)} = e^{c\ln L} = L^c$$

The others can be handled similarly.

$\square$

Theorem 2

Suppose $(a_n)_{n\in\mathbb{Z}^+}$ is a sequence such that $\lim_{n\to\infty} a_n$ exists. $\forall c \in \mathbb{R}$,

$$\lim_{n\to\infty} a_n^c \quad = \quad \left(\lim_{n\to\infty} a_n\right)^c \quad \text{provided that } \lim_{n\to\infty} a_n > 0$$

Proof

Suppose $\lim_{n\to\infty} a_n = L > 0$.

$$\exists N \in \mathbb{Z}^+, \forall n > N, \quad |a_n - L| < \frac{L}{2} \implies a_n > \frac{L}{2} > 0$$

So a_n^c is well-defined for sufficiently large n. By continuity of $e^x, \ln x$,

$$\lim_{n\to\infty} a_n^c = \lim_{n\to\infty} e^{c\ln a_n} = e^{c\ln L} = L^c$$

$\square$

Theorem 3

"Elementary functions" are all continuous on their domains.
(x^a, polynomials, rational functions, $a^x, \log_a x$, trigonometric functions)

Proof

$f(x) = a^x$ with $D_f = \mathbb{R}$ where $a > 0$ and $a \neq 1$

Since $a^x = e^{x\ln a}$, the continuity of $f(x)$ follows from the continuity of e^x and $(\ln a)x$.

$f(x) = \log_a x$ with $D_f = (0, \infty)$ where $a > 0$ and $a \neq 1$

Since $\log_a x = \dfrac{\ln x}{\ln a}$ where $\ln a \neq 0$, the continuity of $f(x)$ follows from the continuity of $\ln x$ and $\dfrac{x}{\ln a}$.

$f(x) = x^a$ where $a \neq 0$

When $x > 0$,

$$f(x) = x^a = e^{a\ln x}$$

is continuous at x.

When $x < 0$, we must have $a = \dfrac{p}{q}$ for some $p, q \in \mathbb{Z}$ where q is odd. Then,

$$f(x) = x^{\frac{p}{q}} = (\sqrt[q]{x})^p = (-\sqrt[q]{(-x)})^p = (-1)^p(-x)^a$$

Thus, f is continuous at x.

At $x = 0$, we must have $a > 0$. By continuity of $e^x, \ln x$ and ax,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{a\ln x} = e^{-\infty} = 0 = f(0)$$

If $D_f = \mathbb{R}$, then we must have $a = \dfrac{p}{q}$ for some $p, q \in \mathbb{Z}$ where q is odd.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-1)^p(-x)^a = \lim_{y\to 0^+} (-1)^p y^a = 0 = f(0)$$

Hence, f is continuous at 0.

$f(x)$ is a polynomial with $D_f = \mathbb{R}$

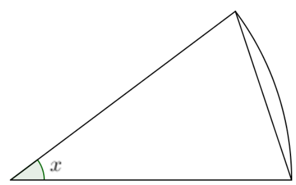
Follows from the fact that constant functions and x^n with $n \in \mathbb{Z}^+$ are continuous.

$f(x)$ is a rational function

Simply because $f(x) = \dfrac{p(x)}{q(x)}$ where $p(x), q(x)$ are polynomials.

$f(x)$ is a trigonometric function

Clearly, it's enough to show the continuity of $\sin x, \cos x$ over $\mathbb{R}$.



radius = 1

By comparing areas, we can see that, for all $x \in (0, \frac{\pi}{2})$,

$$0 \le \frac{1}{2}\sin x \le \frac{1}{2}x$$

By squeeze theorem, $\lim\limits_{x\to 0^+} \sin x = 0 = \sin 0$. Moreover,

$$\lim_{x\to 0^-} \sin x = \lim_{y\to 0^+} \sin(-y) = \lim_{y\to 0^+} (-\sin y) = 0$$

Thus, $\sin x$ is continuous at 0. Furthermore,

$$\lim_{x\to 0} \cos x = \lim_{x\to 0} \sqrt{1 - \sin^2 x} = 1 = \cos 0$$

So, $\cos x$ is continuous at 0. Hence,

$$\lim_{h\to 0} \sin(x+h) = \lim_{h\to 0}(\sin x \cos h + \cos x \sin h) = \sin x$$
$$\lim_{h\to 0} \cos(x+h) = \lim_{h\to 0}(\cos x \cos h - \sin x \sin h) = \cos x$$

and $\sin x, \cos x$ are continuous over $\mathbb{R}$. □

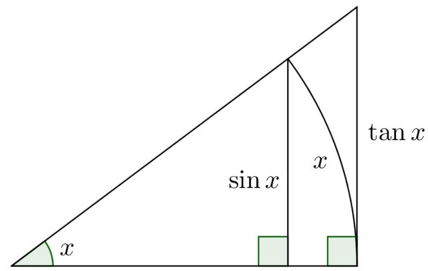
Theorem 4

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

where x is in radian.

Proof

Let us handle $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$ first. When $x > 0$, let's consider the following graph:



radius = 1

By comparing areas, we have

$$\frac{1}{2}\sin x\cos x \leq \frac{1}{2}x \leq \frac{1}{2}\tan x \implies \cos x \leq \frac{x}{\sin x} \leq \frac{1}{\cos x}$$

By continuity of $\cos x$,

$$\lim_{x\to 0^+}\cos x = \lim_{x\to 0^+}\frac{1}{\cos x} = 1,$$

by squeeze theorem, we have $\lim\limits_{x\to 0^+}\frac{x}{\sin x} = 1$.

When $x < 0$, by letting $y = -x$,

$$\lim_{x\to 0^-}\frac{x}{\sin x} = \lim_{y\to 0^+}\frac{-y}{\sin(-y)} = \lim_{y\to 0^+}\frac{y}{\sin y} = 1$$

Hence,

$$\lim_{x\to 0}\frac{\sin x}{x} = \frac{1}{\lim\limits_{x\to 0}\frac{x}{\sin x}} = 1$$

as desired.

Finally,

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x} = \lim_{x\to 0}\left(\sin\frac{x}{2}\cdot\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right) = 0$$

□

Theorem 5: Alternative definition of e^x

For any $x \in \mathbb{R}$,

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

where $n \in \mathbb{Z}^+$.

Proof

First of all,

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right) = 1 \implies 1+\frac{x}{n} > 0 \text{ for sufficiently large } n$$

By L'Hôpital's rule,

$$\begin{aligned}
\lim_{y\to\infty} \ln\left(1+\frac{x}{y}\right)^y &= \lim_{y\to\infty} \frac{\ln\left(1+\dfrac{x}{y}\right)}{y^{-1}} \quad \left(\frac{0}{0}\text{ form}\right) \\
&= \lim_{y\to\infty} \frac{\left(1+\dfrac{x}{y}\right)^{-1}\dfrac{-x}{y^2}}{-y^{-2}} = x
\end{aligned}$$

By sequential criterion and continuity of e^x,

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(1+\frac{x}{n}\right)^n} = e^x$$

□

<u>Theorem 6: First derivative test</u>

Suppose $c \in [a,b]$, f is continuous over $[a,b]$ and f is differentiable, $f' \neq 0$ is continuous over $[a,b] \setminus \{c\}$.

If $c \in (a,b)$ is a critical point of f, then

(a) $f' > 0$ over (a,c) and $f' < 0$ over (c,b)
$\Longrightarrow$ c is a local maximum.

(b) $f' < 0$ over (a,c) and $f' > 0$ over (c,b)
$\Longrightarrow$ c is a local minimum.

(c) Otherwise
$\Longrightarrow$ c is neither a local maximum nor minimum.

If $c = a$, then

(a) $f' < 0$ over (c,b)
$\Longrightarrow$ c is a local maximum.

(b) $f' > 0$ over (c,b)
$\Longrightarrow$ c is a local minimum.

If $c = b$, then

(a) $f' > 0$ over (a,c)
$\Longrightarrow$ c is a local maximum.

(b) $f' < 0$ over (a,c)
$\Longrightarrow$ c is a local minimum.

<u>Proof</u>

<u>$c \in (a,b)$ is a critical point of f</u>

If $f' > 0$ over (a,c) and $f' < 0$ over (c,b), then, by MVT,

$$\forall y \in (a,c), \exists d \in (y,c), \qquad \frac{f(c)-f(y)}{c-y} = f'(d) > 0 \implies f(c) > f(y)$$

$$\forall y \in (c,b), \exists d \in (c,y), \qquad \frac{f(c)-f(y)}{c-y} = f'(d) < 0 \implies f(c) > f(y)$$

Thus, $f(c)$ is the maximum over (a,b).

The argument for local minimum is similar.

If $f' > 0$ over $(a, c) \cup (c, b)$, then, similarly,

$$\forall x \in (a, c), y \in (c, b), \quad f(x) < f(c) < f(y)$$

Then, c is neither local max nor min. If $f' < 0$ over $(a, c) \cup (c, b)$, then, c is neither local max nor min for a similar reason.

Otherwise, for some $x, y \in (a, c)$ or $x, y \in (c, b)$,

$$f'(x) > 0, f'(y) < 0 \implies f'(z) = 0 \text{ for some } z \in (a, c) \cup (c, b)$$

by IVT, which contradicts with our assumption.

$\underline{c = a}$

If $f' < 0$ over (c, b), then, by MVT,

$$\forall y \in (c, b), \exists d \in (c, y), \quad \frac{f(c) - f(y)}{c - y} = f'(d) < 0 \implies f(c) > f(y)$$

Thus, $f(c)$ is the maximum over $[a, b)$.

The argument for local minimum is similar.

Otherwise, for some $x, y \in (c, b)$,

$$f'(x) > 0, f'(y) < 0 \implies f'(z) = 0 \text{ for some } z \in (c, b)$$

by IVT, which contradicts with our assumption.

$\underline{c = b}$

Let $g(x) = f(a + b - x)$. Then, g satisfies the premises over $[a, b]$ and $g(a) = f(b)$.

$$f' > 0 \text{ over } (a, b) \implies g' < 0 \text{ over } (a, b) \implies g(a) \text{ maximum} \implies f(b) \text{ maximum}$$

$$f' < 0 \text{ over } (a, b) \implies g' > 0 \text{ over } (a, b) \implies g(a) \text{ minimum} \implies f(b) \text{ minimum}$$

□

Theorem 7: Second derivative test

Suppose $f(x)$ is differentiable around c, twice-differentiable at c and $f'(c) = 0$. Then,

(a) $f''(c) < 0$
$\implies$ c is a local maximum.

(b) $f''(c) > 0$
$\implies$ c is a local minimum.

Proof

$\underline{f''(c) < 0}$

Assume the contrary:

$$\forall n \in \mathbb{Z}^+, \exists x_n \in \left(c - \frac{1}{n}, c + \frac{1}{n}\right), \quad f(x_n) > f(c)$$

WLOG, say, x_n is an increasing sequence with $\lim_{n\to\infty} x_n = c$. Then,

$$\lim_{x\to c^-} \frac{f'(x) - f'(c)}{x - c} = \lim_{x\to c^-} \frac{f'(x)}{x - c} = f''(c) = L < 0$$

$$\implies \exists\delta > 0, \forall x \in (c-\delta, c), \quad \left|\frac{f'(x)}{x-c} - L\right| < \frac{|L|}{2} \implies \frac{f'(x)}{x-c} < L + \frac{|L|}{2} < 0 \implies f'(x) > 0$$

WLOG, we may assume f is differentiable over $[c-\delta, c]$. By MVT,

$$\exists n \in \mathbb{Z}^+, \quad x_n \in (c-\delta, c) \implies \exists d \in (x_n, c), \quad \frac{f(x_n) - f(c)}{x_n - c} = f'(d) < 0$$

But

$$d \in (c-\delta, c) \implies 0 < f'(d) < 0$$

which is impossible.

$f''(c) > 0$

By considering $-f$. □

Proposition 3: Alternative definition of concavity

If f is differentiable over an interval I, then

$$f \text{ is concave up (down)} \iff f' \text{ is increasing (decreasing)}$$

Proof

$\implies$

For any $x_1, x_2 \in I$ such that $x_1 < x_2$, for all $n \in \mathbb{Z}^+$, let

$$y_n = \left(1 - \frac{1}{2n}\right) x_1 + \frac{1}{2n} x_2$$

Then, $y_n \in (x_1, x_2)$ is a decreasing sequence with $\lim_{n\to\infty} y_n = x_1$. Since f is concave up over $[x_1, x_2]$,

$$\frac{f(y_n) - f(x_1)}{y_n - x_1} \le \frac{\left(1 - \frac{1}{2n}\right) f(x_1) + \frac{1}{2n} f(x_2) - f(x_1)}{\left(1 - \frac{1}{2n}\right) x_1 + \frac{1}{2n} x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

By taking $n \to \infty$, we have

$$f'(x_1) = Rf'(x_1) = \lim_{n\to\infty} \frac{f(y_n) - f(x_1)}{y_n - x_1} \le \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

On the other hand, let

$$z_n = \frac{1}{2n} x_1 + \left(1 - \frac{1}{2n}\right) x_2$$

Then, $z_n \in (x_1, x_2)$ is an increasing sequence with $\lim_{n\to\infty} z_n = x_2$.

$$\frac{f(z_n) - f(x_2)}{z_n - x_2} \geq \frac{\frac{1}{2n}f(x_1) + \left(1 - \frac{1}{2n}\right) f(x_2) - f(x_2)}{\frac{1}{2n}x_1 + \left(1 - \frac{1}{2n}\right) x_2 - x_2} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

By taking $n \to \infty$, we have

$$f'(x_2) = Lf'(x_2) = \lim_{n\to\infty} \frac{f(z_n) - f(x_2)}{z_n - x_2} \geq \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Hence, $f'(x_1) \leq f'(x_2)$.

$\underline{\Longleftarrow}$

Given $x_1, x_2 \in I$ such that $x_1 < x_2$, let

$$F(t) = (1-t)f(x_1) + tf(x_2) - f((1-t)x_1 + tx_2)$$

where $t \in [0, 1]$. It suffices to show $\forall t \in [0, 1], \quad F(t) \geq 0$.

First of all, $F(0) = F(1) = 0$. Also, F is differentiable over $[0, 1]$ with

$$\begin{aligned} F'(t) &= f(x_2) - f(x_1) - (x_2 - x_1)f'((1-t)x_1 + tx_2) \\ &= (x_2 - x_1)\left(\frac{f(x_2) - f(x_1)}{x_2 - x_1} - f'((1-t)x_1 + tx_2)\right) \end{aligned}$$

So, F' is a decreasing function.

Assume $\exists t_0 \in (0, 1), \quad F(t_0) < 0$. Then, by MVT,

$$\exists t_1 \in (0, t_0), \quad 0 > \frac{F(t_0) - F(0)}{t_0 - 0} = F'(t_1)$$

$$\exists t_2 \in (t_0, 1), \quad 0 < \frac{F(1) - F(t_0)}{1 - t_0} = F'(t_2)$$

Thus, $F'(t_1) < 0 < F'(t_2)$, which is a contradiction.

The statement about concave down can be proved by considering $-f$. □

Proposition 4

Suppose $c \in (a, b)$, f is continuous over (a, b) and differentiable over $(a, b) \setminus \{c\}$. If c is a point of inflection of f, then one of the followings holds:

(a) $f''(c) = 0$,

(b) f is not twice differentiable at c.

Proof

Suppose f is twice differentiable at c. Since c is a point of inflection of f, for some $\epsilon > 0$, say, f is concave up on $(c-\epsilon, c)$ and concave down on $(c, c+\epsilon)$. Then, f' is increasing on $(c-\epsilon, c)$ and decreasing on $(c, c+\epsilon)$. Given $x \in (c-\epsilon, c)$,

$$x_n = \left(1 - \frac{1}{n}\right)c + \frac{1}{n}x \in (c-\epsilon, c)$$

defines an increasing sequence that approaches c. Since f' is increasing on $(c-\epsilon, c)$, $f'(x_n)$ is an increasing sequence. Since f' is continuous at c,

$$\lim_{n\to\infty} f'(x_n) = f'(c) \implies f'(x) = f'(x_1) \leq f'(c)$$

In other words, f' is indeed increasing on $(c-\epsilon, c]$. By similar arguments, f' is decreasing on $[c, c+\epsilon)$. Hence,

$$c \text{ is a local maximum of } f' \implies c \text{ is a critical point of } f' \implies f''(c) = 0$$

□