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MATH1510 Calculus for Engineers
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Appendix 3: Natural Exponential Function

3.1 Extended real number system

Definition 1

The **extended real number system**, denoted by $\overline{\mathbb{R}}$, is given by

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$$

with arithmetic operations:
For any $a \in \mathbb{R}$,

$$a + \infty = \infty + a = \infty$$
$$\infty + \infty = \infty$$
$$a - \infty = -\infty + a = -\infty$$
$$-\infty - \infty = -\infty$$
$$\infty - a = \infty$$

$$\infty \cdot \infty = (-\infty) \cdot (-\infty) = \infty$$
$$\infty \cdot a = a \cdot \infty = \begin{cases} \infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0 \end{cases}$$
$$\infty \cdot (-\infty) = (-\infty) \cdot \infty = -\infty$$
$$(-\infty) \cdot a = a \cdot (-\infty) = \begin{cases} -\infty & \text{if } a > 0 \\ \infty & \text{if } a < 0 \end{cases}$$
$$\frac{\infty}{a} = \begin{cases} \infty & \text{if } a > 0 \\ -\infty & \text{if } a < 0 \end{cases}$$
$$\frac{-\infty}{a} = \begin{cases} -\infty & \text{if } a > 0 \\ \infty & \text{if } a < 0 \end{cases}$$
$$\frac{a}{\infty} = \frac{a}{-\infty} = 0$$

and order $-\infty < a < \infty$.

Theorem 1: Sequential criterion for continuity (extended)

For any function $f(x)$ with domain, codomain $\subseteq \overline{\mathbb{R}}$ and $a \in D_f$,

$$f(x) \text{ is continuous at } a \iff$$

$$\forall (a_n)_{n\in\mathbb{Z}^+} \text{ such that } a_n \in \mathbb{R} \text{ and } \lim_{n\to\infty} a_n = a, \quad \lim_{n\to\infty} f(a_n) = f(a)$$

Proof

Let $L = f(a) \in \overline{\mathbb{R}}$.

$\Longleftarrow$

By the sequential criterion of limits.

$\Longrightarrow$

The case when $a, L \in \mathbb{R}$ were handled before. The cases when $a = \pm\infty$ follow directly from sequential criterion on limits.

When $a \in \mathbb{R}, L = \infty$, given $(a_n)_{n\in\mathbb{Z}^+}$ such that $a_n \in \mathbb{R}$ and $\lim_{n\to\infty} a_n = a$, for all $M \in \mathbb{R}$,

$$f \text{ continuous at } a \implies \exists\delta > 0, \forall x \text{ such that } 0 < |x - a| < \delta, \quad f(x) > M$$

$$f(a) = L = \infty \implies \forall x \text{ such that } |x - a| < \delta, \quad f(x) > M$$

Since $\lim_{n\to\infty} a_n = a$, we have

$$\exists N \in \mathbb{Z}^+, \forall n > N, \quad |a_n - a| < \delta \implies f(a_n) > M$$

Hence, $\lim_{n\to\infty} f(a_n) = \infty = f(a)$. The proof when $a \in \mathbb{R}, L = -\infty$ is similar. □

Proposition 1

(a) Suppose f, g are functions with domains, codomains $\subseteq \overline{\mathbb{R}}$. If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

(b) Constant functions and $f(x) = cx$, where $c \in \mathbb{R}, c \neq 0$, are continuous over $\overline{\mathbb{R}}$.

(c) $g(x) = \dfrac{1}{x}$ is continuous at ∞.

Proof

(a) Given a sequence $(a_n)_{n\in Z^+}$ such that $a_n \in \mathbb{R}$ and $\lim_{n\to\infty} a_n = a$, by the sequential criterion of continuity,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} f(a_n) = f(a) = b$$

If $b_n \in \mathbb{R}$ for sufficiently large n, then we also have

$$\lim_{n\to\infty} g(b_n) = g(b)$$

Hence,

$$\lim_{n\to\infty} (g \circ f)(a_n) = \lim_{n\to\infty} g(b_n) = g(b) = (g \circ f)(a) \implies g \circ f \text{ is continuous at } a$$

Otherwise, there exists subsequence $(b_{n_k})_{k\in\mathbb{Z}^+}$ such that $b_{n_k} = \pm\infty$. Notice that

$$b \in \mathbb{R} \implies \forall\epsilon > 0, \exists N \in \mathbb{Z}^+, \forall n > N, \quad |b_n - b| < \epsilon \quad \text{(contradiction)}$$

Thus, $b = \pm\infty$. Suppose $b = \infty$. Since $\lim_{n\to\infty} b_n = b$, $b_n \neq -\infty$ for sufficiently large n. By continuity of g at b, for sufficiently large n,

$$g(f(a_n)) = \begin{cases} g(b) & \text{if } b_n = \infty \\ g(b_n) \to g(b) & \text{if } b_n \in \mathbb{R} \end{cases}$$

Hence, $\lim_{n\to\infty} (g \circ f)(a_n) = g(b) = (g \circ f)(a)$ and $g \circ f$ is continuous at a. The proof when $b = -\infty$ is similar.

(b) Let $g(x)$ be a constant function, which is continuous over $\mathbb{R}$.

$$\forall \epsilon > 0, \text{ take } N = 1, \forall x > N, \quad |g(x) - g(\infty)| = 0 < \epsilon$$

So, $\lim_{x\to\infty} g(x) = g(\infty)$ and g is continuous at ∞. Similarly, g is also continuous at $-\infty$

Let $f(x) = cx$, which is continuous over $\mathbb{R}$. Suppose $c > 0$.

$$\forall M \in \mathbb{R}, \text{ take } N = \frac{M}{c}, \forall x > N, \quad f(x) = cx > cN = M$$

So, $\lim_{x\to\infty} f(x) = \infty = f(\infty)$ and f is continuous at ∞. Similarly, f is also continuous at $-\infty$. The arguments when $c < 0$ are also similar.

(c)

$$\forall \epsilon > 0, \text{take } N = \frac{1}{\epsilon}, \forall x > N, \quad x > \frac{1}{\epsilon} \implies |g(x) - g(\infty)| = \frac{1}{x} < \epsilon$$

□

3.2 Natural exponential function

Definition 2

The **natural exponential function**, denoted by e^x, is defined by

$$e^x = 1 + \sum_{k=1}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and $e^{-\infty} = 0, e^{\infty} = \infty$.

Theorem 2

The natural exponential function is well-defined for any $x \in \overline{\mathbb{R}}$. Moreover,

$$e^0 = 1, \quad (\forall x \in \mathbb{R}, \quad e^x \in \mathbb{R}) \quad \text{and} \quad (\forall x \in \mathbb{R}^+, \quad e^x > 1)$$

Proof

(In fact, convergence can be proved easily by ratio test)

Clearly, $e^0 = 1$ and we may assume $x \in \mathbb{R}$.

When $x > 0$, let

$$a_n = 1 + \sum_{k=1}^{n} \frac{x^k}{k!}$$

Then, $(a_n)_{n\in\mathbb{Z}^+}$ is an increasing sequence. Let $m = \lfloor x \rfloor + 1$.

$\forall n > m$,

$$\begin{aligned}
a_n &\leq 1 + \frac{m}{1!} + \frac{m^2}{2!} + \cdots + \frac{m^n}{n!} \\
&= 1 + \frac{m}{1!} + \cdots + \frac{m^m}{m!} + \frac{m^{m+1}}{(m+1)!} + \cdots + \frac{m^n}{n!} \\
&= \left(1 + \frac{m}{1!} + \cdots + \frac{m^m}{m!}\right) + \frac{m^m}{m!}\left(\frac{m}{m+1} + \frac{m^2}{(m+2)(m+1)} + \cdots + \frac{m^{n-m}}{n(n-1)\cdots(m+1)}\right) \\
&\leq \left(1 + \frac{m}{1!} + \cdots + \frac{m^m}{m!}\right) + \frac{m^m}{m!}\left(r + r^2 + \cdots + r^{n-m}\right) \quad \text{where } r = \frac{m}{m+1} \\
&\leq \left(1 + \frac{m}{1!} + \cdots + \frac{m^m}{m!}\right) + \frac{m^m}{m!} \cdot \frac{r}{1-r}
\end{aligned}$$

Therefore, a_n is bounded above for sufficiently large n. By monotone convergence theorem,

$$e^x = \lim_{n\to\infty}\left(1 + \sum_{k=1}^{n} \frac{x^k}{k!}\right) = \lim_{n\to\infty} a_n$$

exists in $\mathbb{R}$. Moreover, $e^x \geq a_1 = 1 + x > 1$.

When $x < 0$, let

$$b_n = (1+x) + \sum_{k=1}^{n}\left(\frac{x^{2k}}{(2k)!} + \frac{x^{2k+1}}{(2k+1)!}\right) = (1+x) + \left(\frac{x^2}{2!} + \frac{x^3}{3!}\right) + \cdots + \left(\frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!}\right)$$

Let $m = \lfloor |x| \rfloor + 1$. Since $n > m \implies 2n+1 > 2m+1 > 2|x|$ and

$$\frac{x^{2n}}{(2n)!} + \frac{x^{2n+1}}{(2n+1)!} = \frac{x^{2n}}{(2n)!}\left(1 + \frac{x}{2n+1}\right) > \frac{|x|^{2n}}{(2n)!}\left(1 - \frac{1}{2}\right) > 0$$

as long as $n > m$, the sequence $(b_n)_{n\in\mathbb{Z}^+}$ is increasing for sufficiently large n. Furthermore,

$$b_n \leq (1 + |x|) + \sum_{k=1}^{n}\left(\frac{|x|^{2k}}{(2k)!} + \frac{|x|^{2k+1}}{(2k+1)!}\right) \leq e^{|x|}$$

Hence, by monotone convergence theorem,

$$e^x = \lim_{n\to\infty}\left((1+x) + \sum_{k=1}^{n}\left(\frac{x^{2k}}{(2k)!} + \frac{x^{2k+1}}{(2k+1)!}\right)\right) = \lim_{n\to\infty} b_n$$

exists in $\mathbb{R}$. $\square$

Proposition 2

For all $x, y \in \overline{\mathbb{R}}$ where $x + y$ is well-defined,

$$e^{x+y} = e^x \cdot e^y$$

Proof

The cases when $x = 0$ or $y = 0$, $x = y = \infty$ and $x = y = -\infty$ are trivial. Suppose $x, y \in \mathbb{R}$.

When $x, y > 0$, let

$$\begin{aligned} a_n(x) &= \sum_{k=0}^{n} \frac{x^k}{k!} \\ b_n(y) &= \sum_{k=0}^{n} \frac{y^k}{k!} \\ c_n(x+y) &= \sum_{k=0}^{n} \frac{(x+y)^k}{k!} \end{aligned}$$

For $0 \le i, j \le n$, the coefficient of $x^i y^j$ in $a_n(x) b_n(y)$ is $\dfrac{1}{i! j!}$. On the other hand,

$$\begin{aligned} c_n(x+y) &= \sum_{k=0}^{n} \frac{(x+y)^k}{k!} \\ &= \sum_{k=0}^{n} \left(\frac{C_0^k}{k!} x^k + \frac{C_1^k}{k!} x^{k-1} y + \cdots + \frac{C_k^k}{k!} y^k \right) \end{aligned}$$

That means if $0 \le i + j = k \le n$, then the coefficient of $x^i y^j$ in $c_n(x+y)$ is

$$\frac{C_i^k}{k!} = \frac{k!}{i!(k-i)!k!} = \frac{1}{i!j!}$$

Thus,

$$c_n(x+y) \le a_n(x) b_n(y) \le c_{2n}(x+y)$$

By squeeze theorem,

$$e^x \cdot e^y = \lim_{n \to \infty} a_n(x) b_n(y) = \lim_{n \to \infty} c_n(x+y) = e^{x+y}$$

When $x < 0$ or $y < 0$, terms still agree. So,

$$\begin{aligned} |a_n(x) b_n(y) - c_n(x+y)| &\le \frac{|x|}{1!} \frac{|y|^n}{n!} + \frac{|x|^2}{2!} \left(\frac{|y|^{n-1}}{(n-1)!} + \frac{|y|^n}{n!} \right) + \cdots + \frac{|x|^n}{n!} \left(\frac{|y|}{1!} + \cdots + \frac{|y|^n}{n!} \right) \\ &= a_n(|x|) b_n(|y|) - c_n(|x| + |y|) \end{aligned}$$

By squeeze theorem,

$$\lim_{n \to \infty} |a_n(x) b_n(y) - c_n(x+y)| = 0 \implies e^{x+y} = e^x \cdot e^y$$

In particular, if $x > 0$, then

$$1 = e^0 = e^{x-x} = e^x \cdot e^{-x} \implies e^{-x} = \frac{1}{e^x} \in (0,1)$$

So,

$$\forall y \in \mathbb{R}, \quad e^{\infty+y} = e^{\infty} = \infty = \infty \cdot e^y = e^{\infty} \cdot e^y$$
$$\forall y \in \mathbb{R}, \quad e^{-\infty+y} = e^{-\infty} = 0 = 0 \cdot e^y = e^{-\infty} \cdot e^y$$

and we are done. □

Proposition 3

$$\forall x \in \mathbb{R}, \quad e^x \in (0,\infty) \quad \text{and} \quad e^x \text{ is strictly increasing over } \overline{\mathbb{R}}$$

Proof

By Theorem 2, $e^0 = 1$ and $e^x \in (1,\infty)$ if $x \in \mathbb{R}^+$. Also, as in the proof of Proposition 2, $e^x \in (0,1)$ if $-x \in \mathbb{R}^+$. Therefore,

$$\forall x \in \mathbb{R}, \quad e^x \in (0,\infty)$$

Moreover, for all $x \in \mathbb{R}$,

$$e^{-\infty} = 0 < e^x < \infty = e^{\infty}$$

Finally, whenever $x, y \in \mathbb{R}$ such that $x > y$,

$$e^x = e^{(x-y)+y} = e^{x-y} \cdot e^y > e^y$$

□

Proposition 4

$f(x) = e^x$ is continuous over $\overline{\mathbb{R}}$, differentiable over $\mathbb{R}$ with $f'(x) = f(x)$.

Proof

For all $x \in \mathbb{R}$,

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = e^x \left(\lim_{h\to 0} \frac{e^h - 1}{h} \right)$$

In addition, whenever $|h| < 1$,

$$\begin{aligned} \left| \frac{e^h - 1}{h} - 1 \right| &= \left| \frac{h}{2!} + \frac{h^2}{3!} + \frac{h^3}{4!} + \cdots \right| \\ &\leq |h| \cdot \left| \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \right| = |h| \end{aligned}$$

By squeeze theorem, f is differentiable at x and

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = e^x(1) = f(x)$$

Thus, f is continuous over $\mathbb{R}$. Finally, f is also continuous at $\pm\infty$ because

$$\forall x \in \mathbb{R}^+, \quad e^x \geq x \implies \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^x = \infty = f(\infty)$$

$$\forall y \in \mathbb{R}^+, \quad 0 < e^{-y} = \frac{1}{e^y} < \frac{1}{y} \implies \lim_{x\to -\infty} f(x) = \lim_{y\to\infty} e^{-y} = \lim_{y\to\infty} \frac{1}{y} = 0 = f(-\infty)$$

□

3.3 Exponential and logarithmic functions

Proposition 5

$f:\overline{\mathbb{R}}\to[0,\infty]$ such that $f(x)=e^x$ is bijective. Moreover, f^{-1} is continuous over $[0,\infty]$, differentiable over $(0,\infty)$ with $(f^{-1})'(x)=\dfrac{1}{x}$.

Proof

Given $y\in(0,\infty)$, since f is continuous at $-\infty$,

$$\lim_{x\to-\infty} f(x)=0 \implies \exists N_1\in\mathbb{R},\forall x<N_1,\quad |f(x)|<\frac{y}{2} \implies f(N_1-1)<\frac{y}{2}$$

Similarly, since f is continuous at ∞,

$$\lim_{x\to\infty} f(x)=\infty \implies \exists N_2\in\mathbb{R},\forall x>N_2,\quad f(x)>y+1 \implies f(N_2+1)>y+1$$

Notice that $\dfrac{y}{2}<y<y+1$ and $N_1-1<N_2+1$ because f is strictly increasing. As f is continuous over $\mathbb{R}$, by IVT, there exists $x\in(N_1-1,N_2+1)$ such that $f(x)=y$. So, $f(\mathbb{R})=(0,\infty)$.

Since f is differentiable and $f'(x)=e^x>0$ over $\mathbb{R}$, its inverse f^{-1} exists and is differentiable over $f(\mathbb{R})=(0,\infty)$. Furthermore,

$$(f^{-1})'(x)=\frac{1}{f'(f^{-1}(x))}=\frac{1}{f(f^{-1}(x))}=\frac{1}{x}$$

Naturally, we define

$$f^{-1}(0)=-\infty \quad \text{and} \quad f^{-1}(\infty)=\infty$$

Given $M\in\mathbb{R}$, take $\delta=e^M>0$. Then

$$\forall x \text{ such that } 0<x<\delta,\quad x<\delta=f(M) \implies f^{-1}(x)<M$$

So,

$$\lim_{x\to0^+} f^{-1}(x)=-\infty=f^{-1}(0)$$

Given $M\in\mathbb{R}$, take $N=e^M$. Then

$$\forall x>N,\quad x>N=f(M) \implies f^{-1}(x)>M$$

So,

$$\lim_{x\to\infty} f^{-1}(x)=\infty=f^{-1}(\infty)$$

Hence, f is continuous over $\overline{\mathbb{R}}$. □

Definition 3

$f^{-1}:[0,\infty]\to\overline{\mathbb{R}}$, denoted by $f^{-1}(x)=\ln x$, is the inverse function of $f(x)=e^x$.
For any $a\in\mathbb{R}^+$ and $a\neq1$,

$$a^x=e^{x\ln a} \quad \text{with domain } \overline{\mathbb{R}}$$

$$\log_a x=\frac{\ln x}{\ln a} \quad \text{with domain } [0,\infty]$$

Proposition 6: Basic properties

For any $a \in \mathbb{R}^+$ and $a \neq 1$, $x, y \in \mathbb{R}$,

$$a^{x+y} = a^x \cdot a^y, \qquad a^{x-y} = \frac{a^x}{a^y}, \qquad a^{xy} = (a^x)^y$$

For any $a \in \mathbb{R}^+$ and $a \neq 1$, $x, y \in \mathbb{R}^+$,

$$\log_a(xy) = \log_a x + \log_a y, \qquad \log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y, \qquad \log_a(x^y) = y \log_a x$$

Moreover, for any $a \in \mathbb{R}^+$ and $a \neq 1$, a^x with domain $\mathbb{R}$ and $\log_a x$ with domain $\mathbb{R}^+$ are inverse to each other.

Proof

For any $a \in \mathbb{R}^+$ and $a \neq 1$, $x, y \in \mathbb{R}$,

$$a^{x+y} = e^{(x+y)\ln a} = e^{x \ln a} \cdot e^{y \ln a} = a^x \cdot a^y$$

$$a^x = a^{(x-y)+y} = a^{x-y} \cdot a^y \quad \text{and} \quad a^y = e^{y \ln a} \in (0, \infty) \implies a^{x-y} = \frac{a^x}{a^y}$$

$$(a^x)^y = e^{y \ln(a^x)} = e^{y \ln(e^{x \ln a})} = e^{y(x \ln a)} = a^{xy}$$

For the properties about $\log_a$, it suffices to prove them with $a = e$. For any $x, y \in \mathbb{R}^+$,

$$e^{\ln x + \ln y} = e^{\ln x} \cdot e^{\ln y} = xy \implies \ln x + \ln y = \ln(xy)$$

$$e^{\ln x - \ln y} = \frac{e^{\ln x}}{e^{\ln y}} = \frac{x}{y} \implies \ln x - \ln y = \ln\left(\frac{x}{y}\right)$$

$$e^{y \ln x} = x^y \implies y \ln x = \ln(x^y)$$

For any $a \in \mathbb{R}^+$ and $a \neq 1$, $x \in \mathbb{R}$,

$$a^x = e^{x \ln a} \in \mathbb{R}^+ \quad \text{and} \quad \log_a(a^x) = \frac{\ln a^x}{\ln a} = \frac{x \ln a}{\ln a} = x$$

$\forall x \in \mathbb{R}^+$,

$$\log_a x = \frac{\ln x}{\ln a} \in \mathbb{R} \quad \text{and} \quad a^{\log_a x} = e^{(\log_a x)(\ln a)} = e^{\ln x} = x$$

$\square$