THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
2018 Spring MATH2230
Tutorial 4

Example 1. *Evaluate the integral with the principal branch*

$$\int_C z^{-1+i} dz$$

where C is the positively oriented unit circle.

$$f(z)dz = f(z(\theta))ie^{i\theta}d\theta = e^{(-1+i)i\theta}ie^{i\theta}d\theta = e^{-\theta}id\theta$$
$$\int_C z^{-1+i}dz = \int_{-\pi}^{\pi} e^{-\theta}id\theta = i(-e^{-\pi} + e^{\pi})$$

we should be careful that z^{-1+i} is not defined on the branch cut $\{arg(z) = \pm\pi\}$ but z^{-1+i} is still piecewise continuous on C.

Definition 1. *Suppose C is a contour represented by $z(t) : [a, b] \to \mathbb{C}$, then the length of the contour is the integral*

$$L = \int_a^b |z'(t)|dt$$

Theorem 1. *Suppose C is of length L and $f(z)$ is piecewise continuous on C. If there is a non-negative constant M such that $|f(z)| \leq M$ for all z on C at which $f(z)$ is defined. Then*

$$\left| \int_C f(z)dz \right| \leq ML$$

In $\mathbb{R}^2$, the line integral may be independent of the path taken (only depend on the two ends of the path), we would wonder if it is true for contour integral in $\mathbb{C}$.

Theorem 2. *Suppose that $f(z)$ is continuous in a open connected set D. The following statements are equivalent*

- *f(z) has an antiderivative $F(z)$ throughout D ($F'(z) = f(z)$)*
- *Given two fixed points z_1 and z_2 in D, for any contour lying in D with end points z_1 and z_2, the contour integral has a fixed value depends only on z_1 and z_2*
- *the contour integrals of $f(z)$ around closed contours lying entirely in D all have value zero.*

Moreover, if $f(z)$ has an antiderivative $F(z)$,then

$$\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$$

Remark : We should consider $f(z) = 1/z$. It seems that the antiderivative of f is $F(z) = \log z$, however $\log z$ is not well-defined at the branch cut (in the principal branch, $\log z$ is not well-defined at $arg(z) = \pm\pi$). Hence $f(z) = 1/z$ does not have antiderivative in any open connected set D containing a closed contour. You can compute that $\int_{|z|=1} \frac{dz}{z} = 2\pi i$ which is not zero.

Theorem 3. *(Cauchy-Goursat theorem) If $f(z)$ is analytic at all points interior to and on a simple closed contour C (the closure of the bounded component divided by the contour), then*

$$\int_C f(z)dz = 0$$

It seems that theorem 2 and theorem 3 are quite similar, we may ask if having antiderivative are equivalent to analyticity. Having antiderivative must imply analyticity since we have $F'(z) = f(z)$ in which $F(z)$ is already analytic.

Theorem 4. *A function f that is analytic throughout a simply connected domain U must have an antiderivative everywhere in U and hence $\int_C f(z)dz = 0$ for any closed contour C lying in U.*

Remark : $f = 1/z$ gives a example here to verify that simply connectedness is necessary. You may consider $D = B_1(0) \setminus B_{1/2}(0)$ which is not simply connected but f is analytic here, however f do not have a antiderivative defined on D.

Theorem 5. *(Morera's theorem) A continuous f defined in open connected domain D such that $\int_C f(z)dz = 0$ for any closed contour C lying in D, then $f(z)$ is analytic in D.*

Remark : Actually the statement remains valid for any triangular path C lying in U, we will discuss the reason later.

Exercise:

1 Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant, show that $\left|\int_C \frac{z+5}{z^2-1}dz\right| \leq 7\pi/3$.

2 Use theorem 3 to show that the integrals are zero along the contour $|z| = 1$
(a) $\int_C \frac{dz}{z^2+2z+2}$ (b) $\int_C Log(z+2)dz$.

3 Let C be the of the circle $|z-1| = 2$, compute $\int_C \frac{zdz}{z-1}$.