Mathematical Analysis III

Tutorial 8 (November 14)

The following were discussed in the tutorial this week:

1. Let X be a complete metric space. Let $f : X \to X$ be a continuous map such that f^m is a contraction for some $m \geq 1$. Show that f has a unique fixed point x_0 and $f^n(x)$ converges to x_0 for any $x \in X$.

   Remark. *This is just question 6 in HW 6. We described an alternative method that is different from the suggested solution.*

2. Let U be an open subset of $\mathbb{R}^n$ and $g : U \to \mathbb{R}^n$ be a Lipschitz continuous map with Lipschitz constant α, $0 < \alpha < 1$. Let $f = I + g$, where I is the identity map. Show that

   (a) $f(U)$ is an open set;

   (b) f has an inverse from $f(U)$ to U.

3. Let $K \in C([0, 1] \times [0, 1])$ and $g \in C[0, 1]$. Consider the integral equation

$$\varphi(x) = g(x) + \int_0^1 K(x, y)\varphi^2(y)dy, \quad x \in [0, 1].$$

   Show that the equation has a solution $\varphi \in C[0, 1]$ when g is sufficiently small in the sup-norm.