Solution 9

1. Let Ω be a bounded, convex set in $\mathbb{R}^n$. Show that a family of equicontinuous functions is bounded in $C(\Omega)$ if there exists a point $x_0 \in \Omega$ and a constant $M > 0$ such that $|f(x_0)| \le M$ for all f in the family.

   Solution. By equicontinuity, for $\varepsilon = 1$, there is some δ_0 such that $|f(x) - f(y)| \le 1$ whenever $|x - y| \le \delta_0$. Let $B_R(x_0)$ a ball containing E. Then $|x - x_0| \le R$ for all $x \in E$. We can find $x_0, \cdots, x_n = x$ where $n\delta_0 \le R \le (n+1)\delta_0$ so that $|x_{n+1} - x_n| \le \delta_0$. It follows that
$$|f(x) - f(x_0)| \le \sum_{j=0}^{n-1} |f(x_{j+1} - f(x_j)| \le n \le \frac{R}{\delta_0}.$$
   Therefore,
$$|f(x)| \le |f(x_0)| + n + 1 \le M + \frac{R}{\delta_0} \quad \forall x \in \Omega,\ \forall f \in \mathcal{F}.$$

2. Let $\{f_n\}$ be a sequence in $C(\Omega)$ where Ω is open in $\mathbb{R}^n$. Suppose that on every compact subset of Ω, it is equicontinuous and bounded. Show that there is a subsequence $\{f_{n_j}\}$ converging to some $f \in C(\Omega)$ uniformly on each compact subset of Ω.
   (Hint: Show that $\Omega = \cup_{i=1}^{\infty} K_i$, where K_j are compact subsets of Ω and $K_i \subset K_{i+1}$, for all i.)

   Solution. Let K_j be an ascending family of compact sets in Ω satisfying $\Omega = \bigcup_j K_j$. You may take $K_j = \overline{B_j(0)} \bigcap \{x \in \Omega :\ d(x, \partial\Omega) \ge 1/j\}$. Applying A-A theorem to $\{f_n\}$ on each K_n step by step and then take a Cantor's diagonal sequence.

3. Let $K \in C([a,b] \times [a,b])$ and f_n,$f \in C[a,b]$,define Tf by
$$(Tf)(x) = \int_a^b K(x,y)f(y)dy.$$
   (a) Show that T maps $C[a,b]$ to itself.
   (b) Show that if $\{f_n\}$ is a bounded sequence in $C[a,b]$, then $\{Tf_n\}$ contains a convergent subsequence.

   Solution.

   (a) Since $K \in C([a,b] \times [a,b])$, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|K(x,y) - K(x',y)| < \varepsilon$, whenever $|x - x'| < \delta$. Then for $x, x' \in [a,b], |x - x'| < \delta$, one has
$$|(Tf)(x) - (Tf)(x')| \le \int_a^b |K(x,y) - K(x',y)||f(y)|dy \le |a-b|\|f\|_\infty \varepsilon.$$
   Hence $Tf \in C[a,b]$.

   (b) Suppose $\sup_n \|f_n\|_\infty \le M < \infty$. It follows from the proof of (a) that δ can be taken independent of n. Hence $\{f_n\}$ is equicontinuous. Furthermore, since $|(Tf_n)(x)| \le \int_a^b |K(x,y)||f_n(y)|dy \le M(b-a)\|K\|_\infty$, $\{f_n\}$ is uniformly bounded. Then it follows from Arzela-Ascoli theorem that $\{Tf_n\}$ contains a convergent subsequence.

4. Show that the boundary of a nonempty open set in a metric space must be closed and nowhere dense. Conversely, every closed, nowhere dense set is the boundary of some open set.

Solution. Let U be a nonempty open set and let Γ be its boundary. Then $U \cap \Gamma = \emptyset$, since every point of U is an interior point. Γ is closed since the boundary of a set is always a closed set. Let $x \in \Gamma$. Since x is a boundary point, any metric ball containing x must contain some points in U. It follows that Γ is nowhere dense. Conversely, Let Γ be a closed and nowhere dense set. Let U be the complement of Γ. Then U is open. Let $x \in \Gamma$. Since Γ is nowhere dense, any metric ball containing x must contain some points in U. Hence $\Gamma \subset \partial U$. Since every point of U is an interior point, $\Gamma = \partial U$.

5. Use Baire category theorem to show that transcendental numbers are dense in the set of real numbers.

   Solution. A number is called algebraic if it is a root of some polynomial with integer coefficients and it is transcendental otherwise. Let $\mathcal{A}$ be all algebraic numbers and $\mathcal{T}$ be all transcendental numbers so that $\mathbb{R} = \mathcal{A} \cup \mathcal{T}$. From MATH2050 or even earlier we know that $\mathcal{A}$ is a countable set $\{a_j\}$. Thus let $\mathcal{A}_n = \{a_1, \cdots, a_n\}$ and we have $\mathcal{T} = \cap_n \mathbb{R} \setminus \mathcal{A}_n$. As each $\mathbb{R} \setminus \mathcal{A}_n$ is a dense, open set, $\mathcal{T}$ is a set of second category and therefore dense.

   In case you don't want to use the countability of algebraic numbers, you may let

   $$\mathcal{P}_n = \{ \text{ integer polynomials of degree not exceeding } n \text{ and of coefficients in } \{-n, \ldots, n\}\}$$

   and

   $$\mathcal{B}_n = \{x : \ x \text{ is a root of some polynomials in } \mathcal{P}_n\}.$$

   Then show that each $\mathcal{B}_n$ is closed and nowhere dense. Therefore, $\mathcal{A} = \cup_n \mathcal{B}_n$ is of first category. $\mathcal{B}_n$ is closed since $\mathcal{P}_n$, and hence $\mathcal{B}_n$, is finite. To show nowhere dense of $\mathcal{B}_n$, you may assume the existence of at least one transcendental number α, say. Then for every algebraic number a, show that $a + n^{-1}\alpha$ is a transcendental number so you can always find a transcendental number no matter how close to a.

   A final remark is, while it is easy to show transcendental numbers are dense, here we show that it is of second category, a bit more information.