Solution 8

1. Let $f : \mathbb{R} \to \mathbb{R}$ be C^2 and $f(x_0) = 0, f'(x_0) \neq 0$. Show that there exists $\rho > 0$ such that

$$Tx = x - \frac{f(x)}{f'(x)}, \quad x \in (x_0 - \rho, x_0 + \rho),$$

is a contraction. (This is the Newton's method.)

Solution. By direct computation, $T'(x) = \dfrac{f(x)f''(x)}{f'(x)^2}$. Since f is C^2 and $f(x_0) = 0, f'(x_0) \neq 0$, it follows that T is C^1 in a neighbourhood of x_0 with $T(x_0) = x_0, T'(x_0) = 0$ and hence there exists $\rho > 0$, $\gamma \in (0,1)$ such that

$$|T'(x)| \leq \gamma < 1, \quad \text{for any } x \in [x_0 - \rho, x_0 + \rho].$$

Now mean value theorem implies that

$$|T(x) - T(y)| \leq \gamma|x - y|, \quad \text{for all } x, y \in [x_0 - \rho, x_0 + \rho].$$

In particular, taking $y = x_0$, we have

$$|T(x) - x_0| = |T(x) - T(x_0)| < |x - x_0| \leq \rho, \quad \text{for any } x \in [x_0 - \rho, x_0 + \rho].$$

Hence T is a contraction on $[x_0 - \rho, x_0 + \rho]$.

2. Let $g : U \to \mathbb{R}^n$ be a Lipschitz continuous map on an open set $U \subset \mathbb{R}^n$ with Lipschitz constant α satisfying $0 < \alpha < 1$. Let $f = I + g$, where I is the identity on $\mathbb{R}^n$.

Show that
(a) $f(U)$ is an open set
(b) f has an inverse from $f(U)$ to U.

Solution.
(a) Setting as in the hint, let $B_\delta(x_0) \subset U$. We claim that there is some ρ so that T maps $\overline{B_\delta(0)}$ to itself for all $y \in B_\rho(0)$. For, $Tx = x - (\widetilde{f}(x) - y) = y + g(x_0) - g(x + x_0)$, and we have

$$|Tx| \leq |y| + |g(x_0) - g(x + x_0)| \leq \rho + \alpha|x| \leq \delta \ ,$$

provided we choose $\rho \leq (1 - \alpha)\delta$. So T is a continuous map from $\overline{B_\delta(0)}$ to itself. Next, we have $|Tx - Tz| = |g(x + x_0) - g(z + x_0)| \leq \alpha|x - z|$, so T is a contraction. Since $\overline{B_\delta(0)}$ is a closed set in the complete space $\mathbb{R}^n$, it is also complete. By Banach Fixed Point Theorem we obtain a fixed point x^* for T. From $Tx^* = x^*$, we get $f(x^* + x_0) = \widetilde{f}(x^*) + f(x_0) = y + y_0$, that is, for every $y_1 \in B_\rho(y_0)$, there is a unique point $x_1 \equiv x^* + x_0$ in $B_\delta(x_0)$ satisfies $f(x_1) = y_1$. We have shown that $f(U)$ is open.

(b) Let $f(x_1) = f(x_2)$. Then $x_1 + g(x_1) = x_2 + g(x_2)$ implies

$$|x_1 - x_2| = |g(x_2) - g(x_1)| \leq \alpha|x_2 - x_1|,$$

which forces $x_1 - x_2 = 0$. Hence f is injective from U onto $f(U)$.

3. Let $A = (a^i_j)_{n \times n}$ be an $n \times n$-matrix with $||A|| = \sqrt{\sum_{i,j} (a^i_j)^2} < 1$

Show that, for all $b \in \mathbb{R}^n$,

$$(I - A)x = b$$

admits a unique solution.

Solution. We define $Tx = x - (I - A)x - b$ on $\mathbb{R}^n$. By Lemma 2.1, $\|Tx_2 - Tx_1\|_2 = \|A(x_2 - x_1)\|_2 \leq \|A\|\|x_2 - x_1\|_2$, where $\|A\| = (\sum_{i,j} a_{ij}^2)^{1/2}$. By our assumption $\|A\| < 1$, so T is a contraction on $\mathbb{R}^n$. By the contraction mapping principle, it has a unique fixed point. In other words, the matrix $I - A$ is invertible.

4. Consider the function

$$f(x) = \frac{1}{2}x + x^2 \sin\frac{1}{x}, \quad x \neq 0,$$

and set $f(0) = 0$. Show that f is differentiable at $x = 0$ with $f'(0) = \frac{1}{2}$ but it has no local inverse at 0. Does it contradict the inverse function theorem?

Solution. $|f(x) - f(0) - (1/2)x| = |x^2 \sin(1/x)| = O(x^2)$, hence f is differentiable at 0 with $f'(0) = 1/2$. Let $x_k = 1/2k\pi, y_k = 1/(2k\pi + 1)$, then $f'(x_k) = -1/2, f'(y_k) = 3/2$. Then it is clear that f is not injective in $I_k = (y_k, x_k)$. Since any neighborhood of 0 must include contain some I_k, this shows that f it has no local inverse at 0. It does not contradict the inverse function theorem because $f'(x)$ is not continuous at 0.