Solution 2

1. Show that a function is a trigonometric polynomial if and only if it is a finite Fourier series.

Solution A trigonometric polynomail of degree N is given by

$$f(x) = p(\cos x, \sin x) = \sum_{n=0}^{N}\sum_{k=0}^{n} c_{k,n-k}(\cos x)^k(\sin x)^{n-k},$$

where $p(x, y) = \sum_{n=0}^{N}\sum_{k=0}^{n} c_{k,n-k}x^k y^{n-k}$ is a general polynomial of degree N in two variables x, y.

Suppose $f(x)$ is a trigonometric polynomial of order N, say

$$f(x) = p(\cos x, \sin x) = \sum_{n=0}^{N}\sum_{k=0}^{n} c_{k,n-k}(\cos x)^k(\sin x)^{n-k}.$$

By Euler's formula, $\cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$, one has

$$\begin{aligned} f(x) &= p\left(\frac{e^{ix}+e^{-ix}}{2}, \frac{e^{ix}-e^{-ix}}{2i}\right) \\ &= \sum_{n=0}^{N}\sum_{k=0}^{n} c_{k,n-k}\Big(\frac{e^{ix}+e^{-ix}}{2}\Big)^k\Big(\frac{e^{ix}-e^{-ix}}{2i}\Big)^{n-k} \\ &= \sum_{n=0}^{N}\sum_{k=0}^{n} \frac{c_{k,n-k}2^n}{e^{inx}i^{n-k}}\left[\sum_{r=0}^{k} e^{2irx}\right]\left[\sum_{s=0}^{n-k} e^{2isx}\right] \\ &= \sum_{n=0}^{N}\sum_{m=-n}^{n} d_t e^{imx} \\ &= \sum_{n=-N}^{N} c_n e^{inx}, \end{aligned}$$

by collecting terms in the last two steps. Hence $f(x)$ is a finite Fourier series of order N.

On the other hand, suppose $f(x)$ is a finite Fourier series given by

$$f(x) = \sum_{n=-N}^{N} c_n e^{inx}.$$

Applying Euler's formula and expanding the binomials, we have

$$\begin{aligned} f(x) &= c_0 + \sum_{n=1}^{N}\big(c_n(e^{ix})^n + c_{-n}(e^{-ix})^n\big) \\ &= c_0 + \sum_{n=1}^{N}(c_n(\cos x + i\sin x)^n + c_{-n}(\cos x - i\sin x)^n) \\ &= c_0 + \sum_{n=1}^{N}\left(c_n\sum_{k=0}^{n}\binom{n}{k}(\cos x)^k(i\sin x)^{n-k} + c_{-n}\sum_{k=0}^{n}\binom{n}{k}(\cos x)^k(-i\sin x)^{n-k})\right) \\ &= c_0 + \sum_{n=1}^{N}\sum_{k=0}^{n}\binom{n}{k} i^{n-k}(c_n + (-1)^{n-k}c_{-n})(\cos x)^k(\sin x)^{n-k} \\ &= p(\cos x, \sin x), \end{aligned}$$

where $p(x,y) = c_0 + \sum_{n=1}^{N}\sum_{k=0}^{n}\binom{n}{k} i^{n-k}(c_n + (-1)^{n-k}c_{-n})x^k y^{n-k}$ is a polynomial of degree N in two variables x, y. Hence $f(x)$ is a trigonometric polynomial of degree N.

2. Let f be a 2π periodic function integrable on $[-\pi,\pi]$. Show that $F(x) = \int_0^x f(x)dx$ is 2π-periodic if and only if $\int_{-\pi}^{\pi} f dx = 0$. When this holds, find $a_n(F)$ and $b_n(F)$ in terms of $a_n(f)$ and $b_n(f)$ $\forall n \neq 0$.

   Solution Since $f \in R_{2\pi}$,

$$F(x+2\pi) - F(x) = \int_x^{x+2\pi} f(x)dx = \int_0^{2\pi} f(x)dx.$$

   Hence F is 2π-periodic if and only if f has zero mean. In this case, using Fubini's theorem, one has

$$\begin{aligned}\hat{F}(n) &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^x f(y)e^{-inx}dydx = \frac{1}{2\pi}\int_0^{2\pi}\int_y^{2\pi} f(y)e^{-inx}dxdy \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(y)\frac{1}{-in}(1-e^{-iny})dy = \frac{1}{in}\hat{f}(n), \quad \forall n \neq 0.\end{aligned}$$

   If f is further assumed to be continuous, then using the Fundamental Theorem of Calculus and integration by parts, one can also deduce the same result:

$$\begin{aligned}\hat{F}(n) &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^x f(y)e^{-inx}dydx = -\frac{1}{in2\pi}\int_0^{2\pi}\int_0^x f(y)dyde^{-inx} \\ &= \frac{1}{in2\pi}\int_0^{2\pi} f(x)e^{-inx}dx = \frac{1}{in}\hat{f}(n), \quad \forall n \neq 0.\end{aligned}$$

   Then, $\forall n \geq 1$
   $b_n(F) = i\hat{F}(n) - i\hat{F}(-n) = \frac{i(-i)\hat{f}(n) - i(-i)(-1)\hat{f}(-n)}{n} = \frac{\hat{f}(n)+\hat{f}(-n)}{n} = \frac{a_n(f)}{n}$
   Similarly, $a_n(F) = \hat{F}(n) + \hat{F}(-n) = \frac{-i\hat{f}(n) - i(-1)\hat{f}(-n)}{n} = \frac{-i(\hat{f}(n)-\hat{f}(-n))}{n} = -\frac{b_n(f)}{n}$

3. Let f be a C^∞ 2π-periodic complex-valued function. Show that the (complex) Fourier coefficient $c_n = \circ(n^{-k})$ as $n \to \pm\infty$ for every k.

   Solution A repeated application of Property 2 in Section 1.2 shows that $(in)^{k+1}\hat{f}(n) = \widehat{f^{(k+1)}}(n)$ for every k. The right hand side is uniformly bounded due to the Riemann-Lebsegue Lemma applied to the integrable function $f^{(k+1)}$. It follows that

$$|\hat{f}(n)| \leq \frac{M}{n^{k+1}} = \circ(n^{-k}), \quad M = \sup\{\widehat{f^{k+1}}(n) : \ n \geq 1\}.$$

4. Show that

$$\frac{\pi^2}{12} = \sum_{k=1}^{\infty}\frac{(-1)^{n+1}}{n^2}.$$

   Solution By Homework 1 Q3 a), one has

$$x^2 \sim \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx.$$

Since $x^2 \in R_{2\pi}$ is smooth, by Theorem 1.5, one has

$$0 = \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos 0.$$

Hence

$$\frac{\pi^2}{12} = \sum_{k=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

5. Using Thm 1.6 of Sect 1.3 (in the Notes of Lecture 3), show that for $t \in (0, 1)$,

$$\frac{\pi \cos tx}{\sin t\pi} = \frac{1}{t} + \sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2}(-1)^n \cos nx, \quad x \in [-\pi, \pi].$$

Solution Consider the Fourier series of the function $\cos tx$, $x \in [-\pi, \pi]$, $t \in (0, 1)$
By integration by parts, one has

$$f(x) = \frac{\pi \cos tx}{\sin t\pi} \sim \frac{1}{t} + \sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2}(-1)^n \cos nx.$$

Since f is smooth in $(0, 2\pi)$, and $f(\pi^-) = f(-\pi^+)$ due to $\cos tx$ being an even function, by Theorem 1.6, one has

$$\frac{\pi \cos tx}{\sin t\pi} = \frac{1}{t} + \sum_{n=1}^{\infty} \frac{2t}{t^2 - n^2}(-1)^n \cos nx, \quad x \in [-\pi, \pi].$$