Thm 4.8 (Arzela's Theorem) Let K be a compact subset of $\mathbb{R}^n$. Then a compact subset of $C(K)$ is closed, bounded, and equicontinuous.

Pf: Let $\mathcal{F}$ be a compact subset of $C(K)$.

Then we've shown that $\mathcal{F}$ is closed and bounded.

Since compactness $\Rightarrow$ totally boundedness, we have $\forall \varepsilon > 0$, functions $f_1, f_2, \cdots, f_N \in \mathcal{F}$ such that

$$\mathcal{F} \subset \bigcup_{i=1}^{N} B_\varepsilon(f_i) \quad (\text{where } N = N_\varepsilon)$$

$\therefore \forall f \in \mathcal{F}$, $\exists f_i$, $i = 1, \cdots, N$, such that

$$|f(x) - f_i(x)| \leq \|f - f_i\|_\infty < \varepsilon, \quad \forall x \in K.$$

As f_i is continuous on cpt. K, it is uniformly continuous

$\Rightarrow \exists \delta_i > 0$ such that

$$|f_i(x) - f_i(y)| < \varepsilon, \quad \forall |x-y| < \delta_i \ (x, y \in K)$$

Let $\delta = \min\{\delta_1, \cdots, \delta_N\} > 0$, then $\forall |x-y| < \delta \ (x, y \in K)$,

$$|f(x)-f(y)| \le |f(x)-f_i(x)| + |f_i(x)-f_i(y)| + |f_i(y)-f(y)|$$
$$< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon .$$

\#

Conversely, we have

Thm 4.9 (Ascoli's Theorem) Let K be a compact subset of $\mathbb{R}^n$. Then a closed, bounded, and equicontinuous subset in $C(K)$ is compact.

Pf: Let $\mathcal{F}$ be a closed, bounded and equicontinuous subset in $C(K)$, and $\{f_n\}$ be a sequence in $\mathcal{F}$.

Step 1: Since K is compact in $\mathbb{R}^n$, it is totally bounded

$\Rightarrow \forall\, i = 1, 2, 3, \dots$, $\exists$ finitely many points $x_1^{(i)}, \dots x_{N_i}^{(i)}$ such that

$$K \subset B_{\frac{1}{i}}(x_1^{(i)}) \cup \dots \cup B_{\frac{1}{i}}(x_{N_i}^{(i)}).$$

Then the set $A = \bigcup_{i=1}^{\infty} \{x_1^{(i)}, \dots, x_{N_i}^{(i)}\}$

is countable.

And by the boundedness of $\mathcal{F}$, $\exists M>0$ such that

$$|f_n(x)| \leq M, \quad \forall x\in A \ \& \ n\geq 1.$$

In particular $\quad |f_n(x_k^{(i)})| \leq M, \quad \forall i \ \& \ k=1,\cdots,N_i, \forall n$

<u>Step 2</u> (Lemma 4.10)

<u>Claim</u>: There exists a subsequence of $\{f_n\}$, denoted by $\{g_n\}$, such that $\{g_n(x)\}$ is convergent $\forall x\in A$.

<u>Pf</u>: By step 1, A is countable and hence can be listed as a sequence $A=\{x_1, x_2, x_3, \cdots\}$.

Since $\{f_n(x_1)\}$ is a bounded sequence in $\mathbb{R}$, there exists a subsequence, denoted by $\{f_n^{(1)}\}$, of $\{f_n\}$ such that $\{f_n^{(1)}(x_1)\}$ is convergent.

For x_2, $\{f_n^{(1)}(x_2)\}$ is bounded, $\exists$ subseq. $\{f_n^{(2)}\}$ of $\{f_n^{(1)}\}$ such that $\{f_n^{(2)}(x_2)\}$ is convergent.

Note that $\{f_n^{(2)}\}$ is a subseq. of $\{f_n^{(1)}\}$, $\{f_n^{(2)}(x_1)\}$ is also convergent.

Repeat the process for $x_3, x_4, \cdots, x_k, \cdots$, we will find a sequence of subsequences $\{f_n^{(k)}\}$ of $\{f_n\}$ such that

- $\{f_n^{(k)}\}$ is a subsequence of $\{f_n^{(k-1)}\}$, and
- $\{f_n^{(k)}(x_1)\}, \{f_n^{(k)}(x_2)\}, \cdots, \{f_n^{(k)}(x_k)\}$ are convergent

Then the diagonal trick gives us a subseq

$$\{ g_n = f_n^{(n)} \} \text{ of } \{f_n\}$$

such that $\{g_n(x_i)\}$ is convergent $\forall\, x_i \in A$.

$\left(\{g_n(x_i)\} \text{ is a subseq. of } \{f_n^{(i)}(x_i)\} \right)$. #

<u>Step 3</u> $\{g_n\}$ constructed in Step 2 is a Cauchy sequence in $C(K)$.

<u>Pf</u>: By equicontinuity of $\mathcal{F}$,

$\forall \varepsilon > 0, \exists \delta > 0$ such that

$$|g_n(x) - g_n(y)| < \varepsilon, \ \forall\, |x-y| < \delta \ (x, y \in K)$$

Pick $i_0 > \frac{1}{\delta}$. Then by the construction of the A in Step 1, $\forall x \in K$, $\exists$ some $k = 1, 2, \cdots, N_{i_0}$ such that

$$|x - x_k^{(i_0)}| < \frac{1}{i_0} < \delta.$$

Hence

$$|g_n(x) - g_m(x)| \leq |g_n(x) - g_n(x_k^{(i_0)})| + |g_n(x_k^{(i_0)}) - g_m(x_k^{(i_0)})|$$
$$+ |g_m(x_k^{(i_0)}) - g_m(x)|$$
$$< 2\varepsilon + |g_n(x_k^{(i_0)}) - g_m(x_k^{(i_0)})|$$

By Step 2, $\exists\, n_0 = n_0(k) \geq 1$ such that

$$|g_n(x_k^{(i_0)}) - g_m(x_k^{(i_0)})| < \varepsilon, \ \forall n, m \geq n_0$$

Let $N_0 = \max_{k=1,\cdots,N_{i_0}} \{n_0(k)\} \geq 1$, then

$\forall x \in K$ and $\forall n, m \geq N_0$,

$$|g_n(x) - g_m(x)| < 2\varepsilon + |g_n(x_k^{(i_0)}) - g_m(x_k^{(i_0)})|$$
$$< 3\varepsilon.$$

$\Rightarrow \quad \|g_n - g_m\|_\infty < 3\varepsilon \quad, \forall n, m \geq N_0$

$\therefore \quad \{g_n\}$ is a Cauchy sequence in $C(K)$.

Final Step: By completeness of $C(K)$ and closedness of $\mathcal{F}$, $g_n \to g$ uniformly for some $g \in \mathcal{F}$. Hence $\mathcal{F}$ is compact. ⋕

Thm 4.11 A sequence in $C(K)$, where K is compact in $\mathbb{R}^n$, has a convergent subsequence if it is uniformly bounded and equicontinuous.

Pf: Let $\mathcal{F}$ = closure of $\{f_n\}$.

Since $\{f_n\}$ is uniformly bounded, $\exists M>0$ such that $|f_n(x)| \leq M$, $\forall x \in K$ & $n \geq 1$.

If $f \in \mathcal{F}$, then $f = f_n$ for some n, or $\exists \{f_{n_j}\}$ s.t. $\|f_{n_j} - f\|_\infty \to 0$

$\therefore \quad \|f\|_\infty \leq M, \quad \forall f \in \mathcal{F}.$

$\Rightarrow$ $\mathcal{F}$ is bounded in $C(K)$.

By equicontinuity of $\{f_n\}$, $\forall \varepsilon>0$, $\exists \delta>0$ such that $\forall n \geq 1$, $|f_n(x) - f_n(y)| < \varepsilon$, $\forall\ |x-y| < \delta$ $(x, y \in K)$

If $f \in \mathcal{F}$ and $f \neq f_n$, then $\exists\ f_{n_j} \to f$ in $C(K)$

Taking $n_j \to \infty$ in $|f_{n_j}(x) - f_{n_j}(y)| < \varepsilon$,

we have $|f(x) - f(y)| < \varepsilon$, $\forall\ |x-y| < \delta$ $(x, y \in K)$.

$\therefore$ $\mathcal{F}$ is equicontinuous.

By Arzela-Ascoli's Thm, $\mathcal{F}$ is compact

& hence $\{f_n\}$ has a convergent subsequence. #

Application of Arzela - Ascoli Theorem

Thm 4.12 (Cauchy - Peano Theorem)

Let f be a continuous function on $R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$. Then $\exists\ a' \in (0, a)$ and a C^1-function $x : [t_0 - a', t_0 + a'] \to [x_0 - b, x_0 + b]$ solving the (IVP)

$$\begin{cases} \frac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases}.$$

Pf: <u>Step 0</u>: WLOG, we may assume $t_0 = x_0 = 0$.

<u>Step 1</u>: By Weierstrass approximation theorem, $\exists$ polynomials (of 2 variables) $p_n(t,x)$ approximating f in $C([-a,a]\times[-b,b])$

Since every $p_n(t,x)$ satisfies Lip condition on $[-a,a]\times[-b,b]$ as in Picard-Lindelöf Theorem.

$\therefore$ $\forall n \geq 1$, $\exists$ unique solution x_n, $|x_n| \leq b$, of

$$\begin{cases} \frac{dx}{dt} = p_n(t,x) \\ x(0) = 0 \end{cases}$$

defined on $[-\alpha, \alpha]$ with $\alpha < \min\{a, \frac{b}{M_n}, \frac{1}{L_n}\}$

where $M_n = \max\limits_{[-a,a]\times[-b,b]} |p_n(t,x)|$,

L_n = Lip const. for the Lip condition of $p_n(t,x)$.

<u>Step 2</u>: Use ODE theory to conclude that the unique soln. x_n found in Step 1 actually extends to a soln on the interval

$I_n = (-a_n, a_n)$ where $a_n = \min\{a, \frac{b}{M_n}\}$.

(Pf: Omitted as it is not related to Arzela-Ascoli.)

Step 3: Note that $p_n \to f$ uniformly on $[-a,a]\times[-b,b]$,

we have $M_n \to M = \|f\|_\infty$.

Therefore $\left|\frac{dx_n}{dt}\right| = |p_n(t, x_n)| \le M_n \to M$

$\therefore$ $\{x_n\}$ is a uniformly bounded (as $\|x_n\|_\infty \le b$)

and equicontinuous sequence (uniformly bdd derivatives)

on $[-a', a']$ where $a' < \min\{a, \frac{b}{M}\}$.

Arzela-Ascoli's Thm $\Rightarrow$ $\exists$ subseq $\{x_{n_j}\}$ of $\{x_n\}$

converges uniformly on $[-a', a']$. Moreover, the

subsequence $\{x_{n_j}\}$ converges to a continuous function

$x(t)$ defined on $[-a', a']$ satisfying $x(0)=0$

and $x(t) = \int_0^t f(s, x(s))\,ds$ (as $x_{n_j}(0) = 0$

and $x_{n_j}(t) = \int_0^t p_{n_j}(s, x_{n_j}(s))\,ds$ $\forall n_j$.) #

§4.5 Completeness and Baire Category Theorem

Def: Let (X, d) be a metric space. A subset E is called nowhere dense if its closure does not contain any metric ball.

eg: $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ nowhere dense in $\mathbb{R}$. However,

$\mathbb{Q}$ has empty interior but $\overline{\mathbb{Q}} = \mathbb{R}$ has nonempty interior, so $\mathbb{Q}$ is not nowhere dense.

Notes: (i) Equivalently, E is nowhere dense if $X \setminus \overline{E}$ is dense in X.

(ii) E is nowhere dense in $X \Leftrightarrow \overline{E}$ is nowhere dense in X.

(iii) Every subset of nowhere dense set is nowhere dense.

Thm 4.13 (Baire Category Theorem)

Let $\{E_i\}_{i=1}^{\infty}$ be a sequence of nowhere dense subsets of a <u>complete</u> metric space (X, d). Then $\bigcup_{i=1}^{\infty} \overline{E_i}$

has empty interior.

(Note: $\bigcup_{i=1}^{\infty} \overline{E_i}$ may not be closed, may not be nowhere dense.)