MATH 3060 HW7 Due Date: Nov 16, 2016

1. Let $f: \mathbb{R} \to \mathbb{R}$ be C^2 and $f(x_0)=0$, $f'(x_0)\neq 0$. Show that there exists $\rho > 0$ such that

$$Tx = x - \frac{f(x)}{f'(x)}, \quad x \in (x_0-\rho, x_0+\rho)$$

is a contraction. (This is the Newton's method.)

2. Let $g: U \to \mathbb{R}^n$ be a Lipschitz continuous map on an open set U of $\mathbb{R}^n$ with Lipschitz constant α satisfying $0<\alpha<1$. Let $f = I+g$, where I is the identity on $\mathbb{R}^n$. Show that
   (a) $f(U)$ is an open set;
   (b) f has an inverse from $f(U)$ to U.

3. Let $A = (a^i_j)_{n\times n}$ be an $n\times n$ matrix with

$$\|A\| = \sqrt{\sum_{i,j} (a^i_j)^2} < 1.$$

Show that, for all $b \in \mathbb{R}^n$,

$$(I-A)x = b$$

admits a unique solution.

4. Consider the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{1}{2}x + x^2 \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Shao that f is differentiable at $x=0$ with $f'(0)=\frac{1}{2}$, but it has no local inverse at $x=0$. Does it contradict the inverse function therem?