Thm 3.5 (<u>Implicit Function Theorem</u>)

Let U be an open set in $\mathbb{R}^n \times \mathbb{R}^m$,

$F : U \to \mathbb{R}^m$ is a C^1-map.

Suppose that $(p_0, q_0) \in U$ satisfies $F(p_0, q_0) = 0$,

and $D_y F(p_0, q_0)$ is invertible in $\mathbb{R}^m$.

Then

(1) $\exists$ an open set of the form $V_1 \times V_2 \subset U$ containing (p_0, q_0) and a C^1-map

$$\varphi : \overset{\subset \mathbb{R}^n}{V_1} \longrightarrow \overset{\subset \mathbb{R}^m}{V_2}$$

such that $\quad F(x, \varphi(x)) = 0, \ \forall x \in V_1.$

(2) $\varphi : V_1 \to V_2$ is C^k when F is C^k, $1 \leq k \leq \infty$.

(3) Moreover, if ψ is another C^1-map in some open set Ω containing p_0 to V_2 satisfying

$F(x, \psi(x)) = 0$ and $\psi(p_0) = q_0$, then

$$\psi \equiv \varphi \quad \text{on } \Omega \cap V_1.$$

Recall: If $F = \begin{pmatrix} f^1(x^1, \ldots, x^n, y^1, \ldots, y^m) \\ \vdots \\ f^m(x^1, \ldots, x^n, y^1, \ldots, y^m) \end{pmatrix}$, then

$$D_yF = \begin{pmatrix} \frac{\partial f^1}{\partial y^1} & \cdots & \frac{\partial f^1}{\partial y^m} \\ \vdots & & \\ \frac{\partial f^m}{\partial y^1} & \cdots & \frac{\partial f^m}{\partial y^m} \end{pmatrix}$$

is $m \times m$ & can be regarded as a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^m$.

<u>Pf of Implicit Function Theorem</u>:

Define $\Phi : U^{\subset \mathbb{R}^n \times \mathbb{R}^m} \longrightarrow \mathbb{R}^n \times \mathbb{R}^m$

$$(x,y) \longmapsto (x, F(x,y))$$

where $x = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} \in \mathbb{R}^n$, $y = \begin{pmatrix} y^1 \\ \vdots \\ y^m \end{pmatrix} \in \mathbb{R}^m$.

Clearly Φ is C^k if F is C^k.

In fact

$$\Phi = \begin{pmatrix} x^1 \\ \vdots \\ x^n \\ f^1(x^1, \cdots, x^n, y^1, \cdots, y^m) \\ \vdots \\ f^m(x^1, \cdots, x^n, y^1, \cdots, y^m) \end{pmatrix}$$

$$\Rightarrow D\Phi = \left(\begin{array}{ccc|ccc} 1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & 1 & & & \\ \hline \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} & \frac{\partial f^1}{\partial y^1} & \cdots & \frac{\partial f^1}{\partial y^m} \\ \vdots & & & \vdots & & \vdots \\ \frac{\partial f^m}{\partial x^n} & \cdots & \frac{\partial f^m}{\partial x^n} & \frac{\partial f^m}{\partial y^1} & \cdots & \frac{\partial f^m}{\partial y^m} \end{array}\right)$$

Since $D_y F\Big|_{(p_0,q_0)} = \begin{pmatrix} \frac{\partial f^1}{\partial y^1} & \cdots & \frac{\partial f^1}{\partial y^m} \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial y^1} & \cdots & \frac{\partial f^m}{\partial y^m} \end{pmatrix}$ is invertible in $\mathbb{R}^m$,

$D\Phi\Big|_{(p_0,q_0)}$ is invertible in $\mathbb{R}^n \times \mathbb{R}^m$.

Applying Inverse Function Theorem, $\exists$ local C^1-inverse

$$\Psi = (\Psi_1, \Psi_2) : W^{\subset \mathbb{R}^n \times \mathbb{R}^m} \longrightarrow V,$$

where W and V are open nbds. of $\Phi(p_0,q_0) = (p_0, 0)$ and (p_0,q_0) respectively, and is C^k when F is C^k.

By shrinking the nbds, we may assume V is of the form $V_1 \times V_2$, where V_1 open in $\mathbb{R}^n$ containing p_0; V_2 open in $\mathbb{R}^m$ containing q_0.

Now $\forall\ (x,z) \in W$,

$$\Phi\big(\Psi_1(x,z), \Psi_2(x,z)\big) = (x,z)$$

$$\big(\Psi_1(x,z),\ F(\Psi_1(x,z), \Psi_2(x,z))\big)$$

$\therefore \quad \begin{cases} x = \Psi_1(x,z) \\ z = F(\Psi_1(x,z), \Psi_2(x,z)) \end{cases}$

$\Rightarrow \qquad z = F(x, \Psi_2(x,z))$

In particular, we can take $z=0$ & hence

$$F(x, \Psi_2(x,0)) = 0, \quad \forall x = \Psi_1(x,0) \in V_1.$$

$\therefore \quad \varphi : V_1 \to V_2 : x \mapsto \Psi_2(x,0)$

is the required map s.t.

$$F(x, \varphi(x)) = 0 \quad \text{and is } C^k \text{ when } F \text{ is } C^k.$$

We've proved (1) & (2).

For (3), we observe that the continuity of DF

$\Rightarrow$ we may assume (by shrinking V_1 & V_2 further)

that $\displaystyle\int_0^1 D_yF(x, y_1 + t(y_2 - y_1))\,dt$ is nonsingular

for (x, y_1) & $(x, y_2) \in V_1 \times V_2$.

Now if $\psi : \Omega \to \mathbb{R}^m$ (C^1-map) is another map

s.t. $F(x, \psi(x)) = 0$ & $\psi(p_0) = q_0$,

then $0 = F(x, \psi(x)) - F(x, \varphi(x))$

$$= \left(\int_0^1 D_yF(x, \varphi(x) + t(\psi(x) - \varphi(x)))\,dt\right)(\psi(x) - \varphi(x))$$

($\psi(p_0)=q_0$ guarantees $\psi(x)\in V_2$)

$\therefore \quad \int_0^1 D_y F(x, \varphi(x)+t(\psi(x)-\varphi(x)))\,dt$ nonsingular

$\Rightarrow \quad \psi(x)\equiv\varphi(x), \quad \forall\, x\in \Omega\cap V_1$. #

eg 3.10 : $F:\mathbb{R}^5\to\mathbb{R}^2 : \begin{pmatrix} x\\ y\\ z\\ u\\ v\end{pmatrix} \mapsto \begin{pmatrix} xy^2+xzu+yv^2-3 \\ u^3yz+2xv-u^2v^2-2\end{pmatrix}$

$$D_{\binom{u}{v}}F = \begin{pmatrix} xz & 2yv \\ 3u^2yz-2uv^2 & 2x-2u^2v \end{pmatrix}$$

$$\det D_{\binom{u}{v}}F = xz(2x-2u^2v)-2yv(3u^2yz-2uv^2)$$

$$\det D_{\binom{u}{v}}F \Big|_{\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}} = -2 \neq 0$$

$\therefore$ Implicit Function Thm $\Rightarrow$ $\exists\, \varphi:\mathbb{R}^3\to\mathbb{R}^2$ s.t. $\left(\varphi=\begin{pmatrix}\varphi^1\\ \varphi^2\end{pmatrix}\right)$

$$\varphi\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$$

and
$$F\begin{pmatrix} x\\ y\\ z\\ \varphi^1(x,y,z)\\ \varphi^2(x,y,z)\end{pmatrix}=0 .$$

Note that $\boxed{\text{Inverse Function Thm} \Leftrightarrow \text{Implicit Function Thm}}$

$(\Rightarrow)$ done!

To see ($\Leftarrow$):

For $F : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $F(q_0) = p_0 \in \mathbb{R}^n$, $q_0 \in U$

define
$$\tilde{F} : \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n$$
$$(x, y) \longmapsto x - F(y).$$

Then $\tilde{F}(p_0, q_0) = 0$ $\left(\Leftrightarrow p_0 = F(q_0)\right)$

$D_y \tilde{F} = -DF$ invertible in $\mathbb{R}^n$ (at (p_0, q_0)).

$\therefore$ Implicit Function Thm $\Rightarrow$

$\exists\ \varphi : V_1 \to V_2$ $\left(\begin{array}{l} V_1 \text{ open nbd. of } p_0 \\ V_2 \text{ open nbd. of } q_0 \end{array}\right)$

such that
$$0 = \tilde{F}(x, \varphi(x)) = x - F(\varphi(x))$$

i.e. $\forall x \in V_1$, $\exists\ \varphi(x) \in V_2 \subset \mathbb{R}^n$ s.t.
$$\begin{cases} \varphi(q_0) = p_0 \\ F(\varphi(x)) = x \end{cases}$$

$\therefore$ $\varphi = (F|_{V_2})^{-1}$. (is C^k when F is C^k)

#

§3.4 Picard–Lindelöf Theorem for Differential Equations

Let f be a function defined on

$R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$ where $(t_0, x_0) \in \mathbb{R}^2$

and $a, b > 0$. We consider Cauchy Problem (Initial Value Problem)

$$(3.6) \quad \begin{cases} \dfrac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases}$$

i.e. find a function $x(t)$ defined in a perhaps smaller interval $x : [t_0 - a', t_0 + a'] \to [x_0 - b, x_0 + b]$

such that
$$\begin{cases} x(t) \text{ is differentiable,} \\ x(t_0) = x_0, \text{ and} \\ \dfrac{dx}{dt}(t) = f(t, x(t)), \ \forall t \in [t_0 - a', t_0 + a'] \end{cases}$$

for some $0 < a' \leq a$.

eg 3.11 Consider $\begin{cases} \dfrac{dx}{dt} = 1 + x^2 \\ x(0) = 0 \end{cases}$

Here $f(t, x) = 1 + x^2$ is smooth on $[-a, a] \times [-b, b]$ for any $a, b > 0$. However, the solution $x(t) = \tan t$

defined only on $(-\frac{\pi}{2}, \frac{\pi}{2})$. $\therefore$ Even for nice f, we may still have $a' < a$.

Recall : (i) f defined in $R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$ satisfies the Lipschitz condition (uniform in t) if $\exists\, L > 0$ s.t. $\forall\, (t, x_1), (t, x_2) \in R$,

$$|f(t, x_1) - f(t, x_2)| \leq L |x_1 - x_2| .$$

(ii) In particular, $f(t, \cdot)$ is Lip. cts in x, $\forall\, t \in [t_0 - a, t_0 + a]$.

(iii) L is called a Lipschitz constant.

(iv) If L is a Lip. constant for f, then any $L' > L$ is also a Lip. constant.

(v) Not all cts. functions satisfy the Lip. condition.

eg : $f(t, x) = t x^{1/2}$ is cts, but not Lip. near 0.

(vi) If $f(t, x) : \overset{R}{[t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]} \to \mathbb{R}$ is C^1, then $f(t, x)$ satisfies the Lip. condition:

$$|f(t, x_2) - f(t, x_1)| = \left| \frac{\partial f}{\partial x}(t, y)(x_2 - x_1) \right| \text{ for some } y \in [x_0 - b, x_0 + b]$$

$$\leq L |x_2 - x_1|,$$

where $L = \max \left\{ \left| \frac{\partial f}{\partial x}(t, x) \right| : (t, x) \in R \right\}$.

Thm 3.6 (Picard–Lindelöf Theorem)

Let f be continuous function on $R = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b]$ ($(t_0, x_0) \in \mathbb{R}^2$, $a, b > 0$) satisfies the Lipschitz condition on R. Then $\exists\ a' \in (0, a]$ and $x \in C^1[t_0 - a', t_0 + a']$ such that

$$x_0 - b \le x(t) \le x_0 + b \quad , \quad \forall\ t \in [t_0 - a', t_0 + a']$$

and solving the Cauchy Problem (3.6).

Furthermore, x is the unique solution in $[t_0 - a', t_0 + a']$.

Note: One will see in the following proof that a' can be taken to be any number satisfying

$$0 < a' < \min\left\{a, \frac{b}{M}, \frac{1}{L}\right\}$$

where $M = \sup\{|f(t, x)| : (t, x) \in R\}$ &

L = Lip. const. for f.

Prop 3.7: Setting as in Thm 3.6, every solution x of (3.6) from $[t_0 - a', t_0 + a']$ to $[x_0 - b, x_0 + b]$ satisfies the equation

$$\boxed{x(t) = x_0 + \int_{t_0}^{t} f(t, x(t))\,dt} \qquad (3.7)$$

(Pf: Obvious.)

Proof of Picard-Lindelöf Theorem:

For $a'>0$ to be chosen later, we let

$$X = \{ \varphi \in C[t_0-a', t_0+a'] : \varphi(t_0)=x_0, \ \varphi(t)\in[x_0-b, x_0+b] \}$$

and uniform metric d_∞ on X.

First note that X is closed subset in the complete metric space $(C[t_0-a', t_0+a'], d_\infty)$. Hence

(X, d_∞) is complete.

Define T on X by

$$(T\varphi)(t) = x_0 + \int_{t_0}^{t} f(s,\varphi(s))\,ds$$

(This is well-defined as $\varphi(s)\in[x_0-b, x_0+b]$.)

Clearly $T\varphi \in C[t_0-a', t+a']$ & $(T\varphi)(t_0)=x_0$.

To show $T\varphi\in X$, we need to show that

$$(T\varphi)(t)\in[x_0-b, x_0+b].$$

Let $M=\sup_{(t,x)\in R}|f(t,x)|$ and consider

$$\left|(T\varphi)(t)-x_0\right| = \left|\int_{t_0}^{t} f(s,\varphi(s))ds\right| \le M|t-t_0|$$
$$\le Ma'.$$

So if we choose $0 < a' \leq \frac{b}{M}$, then

$$|(T\varphi)(t) - x_0| \leq b$$

$$\Rightarrow \quad T\varphi \in X.$$

This is, for $0 < a' \leq \frac{b}{M}$, $T: X \to X$ is a self-map from a complete metric space (X, d_∞) to itself.

To see whether T is a contraction, we check

$$\begin{aligned}
|(T\varphi_2 - T\varphi_1)(t)| &= \left|\left(x_0 + \int_{t_0}^{t} f(s, \varphi_2(s))ds\right) - \left(x_0 + \int_{t_0}^{t} f(s, \varphi_1(s))ds\right)\right| \\
&\leq \int_{t_0}^{t} |f(s, \varphi_2(s)) - f(s, \varphi_1(s))| ds \\
&\leq L \int_{t_0}^{t} |\varphi_2(s) - \varphi_1(s)| ds \quad \text{by Lip. condition} \\
&\leq L|t - t_0| \sup_{[t_0 - a', t_0 + a']} |\varphi_2(s) - \varphi_1(s)| \\
&\leq L a' d_\infty(\varphi_2, \varphi_1)
\end{aligned}$$

Therefore, if $La' = \gamma < 1$, then T is a contraction:

$$\begin{aligned}
d_\infty(T\varphi_2, T\varphi_1) &\leq La' d_\infty(\varphi_2, \varphi_1) \\
&= \gamma d_\infty(\varphi_2, \varphi_1).
\end{aligned}$$

In conclusion,

if $0 < a' < \min\{a, \frac{b}{M}, \frac{1}{L}\}$, then

$T: \mathbb{X} \to \mathbb{X}$ is a contraction on a complete metric space. Therefore, by Contraction Mapping Principle, T admits a <u>unique</u> fixed point $x(t) \in \mathbb{X}$.

By Prop 3.7, we've proved Thm 3.6. ※

<u>Note</u>: Existence part of Picard-Lindelöf Thm still holds with $f(t,x)$ cts only (without Lip. condition). However, the solution may not be unique:

<u>eg</u>: Consider $f(t,x) = |x|^{1/2}$ on $\mathbb{R}\times\mathbb{R}$

f is cts, but not Lip. cts.

(Cauchy Problem) $\begin{cases} \frac{dx}{dt} = |x|^{1/2} & \text{on } \mathbb{R} \\ x(0) = 0 \end{cases}$

has solutions $x_1(t) \equiv 0 \quad \forall t \in \mathbb{R}$ and

& $x_2(t) = \begin{cases} \frac{1}{4}t^2, & t \geq 0 \\ -\frac{1}{4}t^2, & t < 0 \end{cases}$

[<u>Check</u>: x_2 is differentiable with $\frac{dx_2}{dt} = \frac{1}{2}|t|, \ \forall t \in \mathbb{R}$
$= |x_2|^{1/2}$

& $x_2(0) = 0$.]