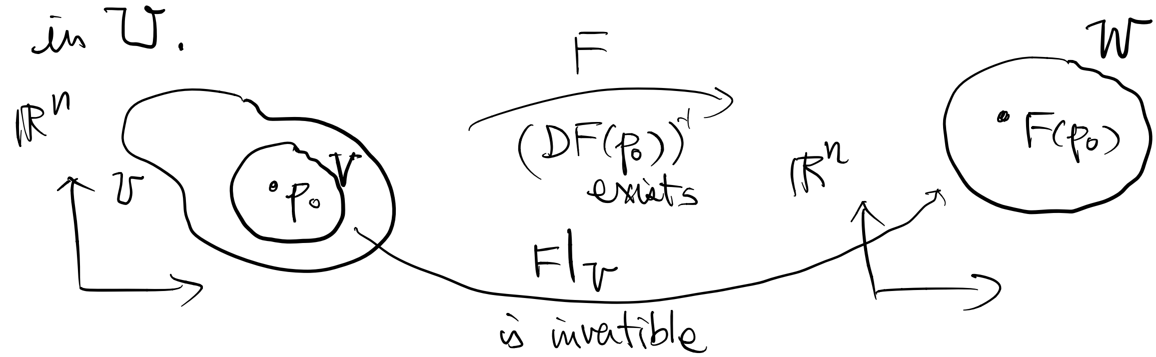
Thm 3.4 (Inverse Function Theorem)

Let $F: U \to \mathbb{R}^n$ be a C^1-map from an open set $U \subset \mathbb{R}^n$. Suppose $p_0 \in U$ and $DF(p_0)$ is invertible (as a matrix or linear transformation). Then $\exists$ open sets V & W containing p_0 and $F(p_0)$ respectively such that the restriction of F on V is a bijection onto W with a C^1-inverse.

Moreover, the inverse is C^k when F is C^k, $1 \le k \le \infty$, in U.



Note: We only have local invertibility by the IFT.

eg 3.5: Let
$$F = (0,\infty) \times (-\infty,\infty) \to \mathbb{R}^2$$
$$(r, \theta) \longmapsto (r\cos\theta, r\sin\theta)$$

Then $DF = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$ invertible $\forall (r,\theta)$

Then IFT $\Rightarrow$ F is locally invertible at every point

$(r,\theta) \in (0,\infty)\times(-\infty,\infty)$. But F is clearly not globally invertible as it is not one-to-one:

$$F(r,\theta+2\pi) = F(r,\theta).$$

eg 3.6 When U = open interval (a,b) in $\mathbb{R}$ $(n=1)$ is a special case: C^1 function $f:(a,b)\to\mathbb{R}$ with $f'\neq 0 \Rightarrow f$ strictly increasing or decreasing $\Rightarrow$ global inverse exists.

eg 3.7: (i) $F:\mathbb{R}^2\to\mathbb{R}^2 : (x,y)\mapsto(x^2,y)$.

$$DF = \begin{pmatrix} 2x & 0 \\ 0 & 1 \end{pmatrix} \text{ singular at } (x,y)=(0,0).$$

F doesn't satisfy the condition DF invertible in the IFT.

And clearly F is not invertible near $(x,y)=(0,0)$ as $F(\pm a,b)=(a^2,b)$ (2-to-1 near $(0,0)$).

(ii) $H:\mathbb{R}^n\to\mathbb{R}^n : (x,y)\mapsto(x^3,y)$ is bijective & $H^{-1}(x,y)=(x^{1/3},y)$ exists.

But $$DH = \begin{pmatrix} 3x^2 & 0 \\ 0 & 1 \end{pmatrix} \text{ singular at } (x,y)=(0,0).$$

The point is:

H^{-1} is not C^1 near $(x,y)=(0,0)$.

Terminology: The condition in IFT that $DF(p_0)$ is invertible is called the nondegeneracy condition.

By eg 3.7, without nondegeneracy condition, the map may or may not local invertible.

Claim: Nondegeneracy condition is necessary for the differentiability of the local inverse.

Pf: Suppose the local inverse F^{-1} exists and is differentiable at the point $q_0 = F(p_0)$. Then

Chain rule $\Rightarrow$ $D(F^{-1})(q_0)\, DF(p_0) = \text{Identity}$

$\Rightarrow$ $DF(p_0)$ is invertible. #

To prove the IFT, we need the following Lemma:

Lemma 3.1 Let $L: \mathbb{R}^n \to \mathbb{R}^n$ be a linear map represented by the matrix (a^i_j) in standard basis,

i.e. $$(Lx)^i = \sum_j a^i_j x^j \quad, i=1,\cdots,n,$$

for all $x = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$, i.e. $L\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} = \begin{pmatrix} a^1_1 & \cdots & a^1_n \\ \vdots & & \vdots \\ a^n_1 & \cdots & a^n_n \end{pmatrix}\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}$

Then $|Lx| \le \|L\|\,|x|$, $\forall x \in \mathbb{R}^n$, where $\|L\| = \sqrt{\sum_{i,j} (a^i_j)^2}$.

<u>Pf</u>: $|Lx|^2 = \sum_i [(Lx)^i]^2$

$$= \sum_i \Big[\sum_j a^i_j x^j \Big]^2$$

$$\le \sum_i \Big[\Big(\sum_j (a^i_j)^2\Big)\Big(\sum_j (x^j)^2\Big)\Big] \quad \text{Cauchy-Schwarz}$$

$$= \Big(\sum_i \sum_j (a^i_j)^2\Big) |x|^2$$

$$= \|L\|^2 |x|^2$$

※

<u>Proof of IFT</u>:

<u>Step 1</u> By considering the new function

$$\tilde{F}(x) = F(x+p_0) - F(p_0),$$

we may assume $p_0 = F(p_0) = 0$ as

$$D\tilde{F}(0) = DF(p_0).$$

Further, by considering a smaller open nbd of p_0, we may assume $DF(x)$ <u>invertible</u> $\forall\, x \in U$.

$(F \in C^1 \Rightarrow \det DF(x)$ cts in $x)$

Step2 : Define, for any fixed y,

$$T^{(y)}x = L^{-1}(Lx - F(x) + y)$$

where $L = DF(0)$.

Then $\exists\ \rho_0 > 0$ and $R > 0$ such that $\forall\ y \in B_R(0)$

$$T^{(y)} : \overline{B_{\rho_0}(0)} \longrightarrow \overline{B_{\rho_0}(0)} \text{ is a self map.}$$

Pf of step2 : By lemma 3.1

$$|Tx| \leq \|L^{-1}\|\ |Lx - F(x) + y| \quad (\text{write } T \text{ for } T^{(y)} \text{ for simplicity})$$

$$\leq \|L^{-1}\| \left[|Lx - F(x)| + |y| \right]$$

F is C^1

$$\Rightarrow F(x) = F(x) - F(0) = \int_0^1 \frac{d}{dt} F(tx)\,dt = \int_0^1 DF(tx)x\,dt$$

$$\therefore\ F(x) - Lx = F(x) - DF(0)x$$

$$= \int_0^1 DF(tx)x\,dt - DF(0)x$$

$$= \left(\int_0^1 [DF(tx) - DF(0)]\,dt \right) x$$

$$|F(x) - Lx| \leq \left\| \int_0^1 [DF(tx) - DF(0)]\,dt \right\| |x|$$

$$\left[\begin{aligned} & A(t) = (a^i_j(t)) \\ & \Rightarrow \int_0^1 A(t) = \left(\int a^i_j\right) \Rightarrow \left\|\int_0^1 A\right\|^2 = \sum_{i,j}\left(\int a^i_j\right)^2 \\ & \qquad\qquad \le \sum_{i,j}\left(\int_0^1 (a^i_j)^2\right)\left(\int_0^1 1^2\right) \\ & \qquad\qquad = \int_0^1 \sum_{i,j}(a^i_j)^2 = \int_0^1 \|A\|^2 \end{aligned}\right]$$

$$\therefore |F(x) - Lx| \le \left[\int_0^1 \|DF(tx) - DF(0)\|^2\,dt\right]^{1/2} |x| .$$

Since DF is cts. at 0, $\exists\, \rho_0 > 0$ such that

$$\|L^{-1}\|\,\|DF(x) - DF(0)\| \le \frac{1}{2}, \quad \forall\, x \text{ with } |x| \le \rho_0$$

$$\Rightarrow \quad |F(x) - Lx| \le \frac{1}{2\|L^{-1}\|}|x|, \quad \forall\, |x| \le \rho_0 .$$

Further choose $R > 0$ such that $\|L^{-1}\|R \le \frac{\rho_0}{2}$.

(ie $R = \frac{\rho_0}{2\|L^{-1}\|} > 0$)

Then $\forall\, y \in B_R(0)$, $|y| \le \frac{\rho_0}{2\|L^{-1}\|}$.

Hence, $\forall\, y \in B_R(0)$

$$|Tx| \le \|L^{-1}\|\left[\frac{1}{2\|L^{-1}\|}\rho_0 + \frac{\rho_0}{2\|L^{-1}\|}\right] = \rho_0 .$$

$\therefore$ $T^{(y)}$ is a self map from $\overline{B_{\rho_0}(0)}$ to itself. $\cancel{\times}$

<u>Step 3</u>: $T^{(y)} : \overline{B_{\rho_0}(0)} \to \overline{B_{\rho_0}(0)}$ is a contraction.

(for $y \in B_R(0)$ & ρ_0 as in step 2)

<u>Pf of Step 3</u>: $\forall\, x_1, x_2 \in \overline{B_{\rho_0}(0)}$, $\quad$ (T for $T^{(y)}$)

$$
\begin{aligned}
|Tx_2 - Tx_1| &= \left|L^{-1}(Lx_2 - F(x_2) + y) - L^{-1}(Lx_1 - F(x_1) + y)\right| \\
&= \left|L^{-1}(Lx_2 - F(x_2) - Lx_1 + F(x_1))\right| \\
&\le \|L^{-1}\| \left|F(x_2) - F(x_1) - DF(0)(x_2 - x_1)\right| \\
&= \|L^{-1}\| \left|\int_0^1 \frac{dF}{dt}(x_1 + t(x_2 - x_1))\,dt - DF(0)(x_2 - x_1)\right| \\
&= \|L^{-1}\| \left|\int_0^1 DF(x_1 + t(x_2 - x_1))(x_2 - x_1)\,dt - DF(0)(x_2 - x_1)\right| \\
&= \|L^{-1}\| \left|\int_0^1 [DF(x_1 + t(x_2 - x_1)) - DF(0)]\,dt\,(x_2 - x_1)\right| \\
&\le \|L^{-1}\| \left\|\int_0^1 [DF(x_1 + t(x_2 - x_1)) - DF(0)]\,dt\right\| |x_2 - x_1| \\
&\le \|L^{-1}\| \left(\int_0^1 \|DF(x_1 + t(x_2 - x_1)) - DF(0)\|^2\,dt\right)^{1/2} |x_2 - x_1|
\end{aligned}
$$

Note $x_1, x_2 \in \overline{B_{\rho_0}(0)} \Rightarrow x_1 + t(x_2 - x_1) \in \overline{B_{\rho_0}(0)}$, $\forall\, t \in [0,1]$

$$\Rightarrow \quad \|L^{-1}\| \, \|DF(x_1+t(x_2-x_1)) - DF(0)\| \leq \frac{1}{2}, \ \forall t \in [0,1].$$

$$\therefore \quad |Tx_2 - Tx_1| \leq \frac{1}{2}|x_2 - x_1|, \ \forall x_1, x_2 \in \overline{B_{\rho_0}(0)}.$$

$\therefore \quad T: \overline{B_{\rho_0}(0)} \to \overline{B_{\rho_0}(0)}$ is a contraction.

Step 4: F is invertible near 0

Pf of Step 4: By contraction mapping principle, we have for any $y \in B_R(0)$, $\exists$ unique fixed point of

$$T^{(y)}: \overline{B_{\rho_0}(0)} \to \overline{B_{\rho_0}(0)},$$

i.e. $\exists!\ x \in \overline{B_{\rho_0}(0)}$ such that

$$x = T^{(y)}x = L^{-1}(Lx - F(x) + y)$$

$$\Rightarrow \quad Lx = Lx - F(x) + y$$

$$\Rightarrow \quad F(x) = y.$$

Since $y \in B_R(0)$ is arbitrary, we have constructed a map

$$G: B_R(0) \to \overline{B_{\rho_0}(0)} \subset U$$

$$y \mapsto x = \text{unique fixed pt. of } T^{(y)},$$

such that $\quad F(G(y)) = y.$

Note that $G(0)=0$ by uniqueness of fixed point.

Let $V = G(B_R(0))^{c\mathcal{U}}$ containing 0, then

$$F\big|_V : V \to B_R(0)$$

is invertible. This in turns implies V is open as F is cts.

<u>Step 5</u>: $G = (F|_V)^{-1}$ is continuous.

<u>Pf of step 5</u>: $|G(y_2)-G(y_1)| = |x_2-x_1|$

$$= |T^{(y_2)}x_2 - T^{(y_1)}x_1|$$

$$= |L^{-1}(Lx_2-F(x_2)+y_2) - L^{-1}(Lx_1-F(x_1)+y_1)|$$

$$\le \|L^{-1}\|\Big[|L(x_2-x_1)-F(x_2)+F(x_1)| + |y_2-y_1|\Big]$$

As in step 3, $\|L^{-1}\|\,|L(x_2-x_1)-F(x_2)+F(x_1)| \le \frac{1}{2}|x_2-x_1|$

$$\forall\, x_1, x_2 \in \overline{B_{\rho_0}(0)}.$$

$$\therefore\ |G(y_2)-G(y_1)| \le \frac{1}{2}|x_2-x_1| + \|L^{-1}\|\,|y_2-y_1|$$

$$= \frac{1}{2}|G(y_2)-G(y_1)| + \|L^{-1}\|\,|y_2-y_1|$$

$$\Rightarrow\quad |G(y_2)-G(y_1)| \le 2\|L^{-1}\|\,|y_2-y_1| \qquad (*)$$

$\therefore$ G is (Lip) cts. on $B_R(0)$.

Step6: $G=(F|_V)^{-1}$ is C^k on $B_R(0)$ if $F\in C^k$.

Pf of Step6: Consider $\Delta G = G(y+y_0)-G(y_0)$.

Note that

$$F(G(y+y_0)) - F(G(y_0)) = y+y_0-y_0 = y$$

$$\begin{aligned}\therefore\ y &= F(G(y+y_0))-F(G(y_0))\\ &= \int_0^1 \frac{dF}{dt}\left[G(y_0)+t\left(G(y+y_0)-G(y_0)\right)\right]dt\\ &=\left(\int_0^1 DF\left[G(y_0)+t\Delta G\right]dt\right)\Delta G\\ &= \left[\int_0^1 DF(G(y_0))\,dt + \int_0^1 \left[DF\left[G(y_0)+t\Delta G\right]-DF(G(y_0))\right]dt\right]\Delta G\\ &= DF(G(y_0))\Delta G + A\Delta G,\end{aligned}$$

where $A=\int_0^1 \left[DF(G(y_0)+t\Delta G)-DF(G(y_0))\right]dt$

Since $F\in C^1$, DF is cts. Together with G is cts (step 5)

$$\|A\| \le \left(\int_0^1 \|DF(G(y_0)+t\Delta G)-DF(G(y_0))\|^2 dt\right)^{1/2} \to 0$$

as $y\to 0$.

$\therefore \quad y = DF(G(y_0))\Delta G + A\Delta G$

$\Rightarrow \quad \Delta G = (DF)^{-1}(G(y_0))y - (DF)^{-1}(G(y_0))A\,\Delta G$

Now $\quad |DF^{-1}(G(y_0))A\Delta G| \le \|DF^{-1}(G(y_0))\|\,|A\Delta G|$

$$\le \|DF^{-1}(G(y_0))\|\,\|A\|\,|\Delta G|$$

(by $(*)$ in step 5) $\le \|DF^{-1}(G(y_0))\|\,\|A\|\cdot 2\|L^{-1}\|\,|y|$

$$= o(1)|y| = o(|y|).$$

$\therefore \quad G(y+y_0) - G(y_0) - (DF)^{-1}(G(y_0))y = o(|y|)$

$\therefore \quad DG(y_0)$ exists & $DG(y_0) = (DF)^{-1}(G(y_0))$

$\forall y_0 \in B_R(0)$.

By Linear algebra, entries of $DG(y_0)$ can be expressed as a rational functions of the entries of $DF(G(y_0))$ (with $\det DF(G(y_0)) \neq 0$ as denominator),

$\therefore \quad F$ is $C^1 \Rightarrow G$ is C^1.

Then repeated applications of the same argument implies F is $C^k \Rightarrow G$ is C^k ※

This complete the proof of IFT.