Thm 3.3 (Contraction Mapping Principle)

Every contraction in a complete metric space admit a unique fixed point.

(This is also called the Banach's Fixed Point Thm)

Pf: Uniqueness: Suppose x & y are fixed pts. of T. Then

$$d(x,y) = d(Tx, Ty) \quad (x, y \text{ are fixed by } T)$$

$$\leq \gamma\, d(x,y) \quad \text{for some } \gamma \in (0,1). \quad (T \text{ contraction})$$

$$\Rightarrow d(x,y) = 0 \Rightarrow x = y.$$

Existence: Let $x_0 \in X$.

Define $\{x_n\}_{n=1}^{\infty}$ by $x_n = Tx_{n-1}$, for $n = 1, 2, \cdots$

Then $x_n = Tx_{n-1} = T(Tx_{n-2}) = T^2 x_{n-2}$

$$= \cdots = T^n x_0.$$

For any $n \geq N$,

$$d(x_n, x_N) = d(T^n x_0, T^N x_0) = d(T^{(n-N)+N} x_0, T^N x_0)$$

$$= d(T(T^{(n-N)+N-1} x_0), T(T^{N-1} x_0))$$

$$\leq \gamma\, d(T^{(n-N)+N-1}x_0, T^{N-1}x_0)$$

(where $\gamma \in (0,1)$ is the constant s.t. $d(Tx,Ty) \leq \gamma d(x,y)$, $\forall x,y \in X$)

$$\leq \cdots$$

$$\leq \gamma^N d(T^{(n-N)}x_0, x_0)$$

$$\leq \gamma^N \Big[d(T^{(n-N)}x_0, T^{(n-N)-1}x_0) + d(T^{(n-N)-1}x_0, T^{(n-N)-2}x_0) + \cdots + d(Tx_0, x_0) \Big]$$

$$\leq \gamma^N \Big[d(Tx_0,x_0) + \gamma d(Tx_0,x_0) + \cdots + \gamma^{(n-N)-2} d(Tx_0,x_0) + \gamma^{(n-N)-1} d(Tx_0,x_0) \Big]$$

$$= \gamma^N \left[1 + \gamma + \cdots + \gamma^{(n-N)-1} \right] d(Tx_0\, x_0)$$

$$< \frac{\gamma^N}{1-\gamma} d(Tx_0, x_0)$$

Therefore, $\forall \varepsilon > 0$, if $N > 0$ is chosen s.t.

$$\frac{\gamma^N}{1-\gamma} d(Tx_0, x_0) < \frac{\varepsilon}{2},$$

we have $\forall\, n, m \geq N$,

$$d(x_n, x_m) \leq d(x_n, x_N) + d(x_N, x_m)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

$\therefore \{x_n\}$ is a Cauchy seq. in (X, d).

By completeness of (X, d), $\exists\, x \in X$ s.t.

$$x_n \to x .$$

Note that a contraction is always continuous (Ex!)

we have

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} T x_{n-1} = T \lim_{n\to\infty} x_{n-1} = Tx .$$

$\therefore$ x is a fixed point of T. ⋕

eg 3.4 Let $f : [0,1] \to [0,1]$ continuously differentiable with $|f'(x)| < 1$ on $[0,1]$. Then f has a fixed point in $[0,1]$.

Pf: By mean value theorem

$\forall\, x, y \in [0,1]$, $\exists\, z \in [0,1]$ s.t.

$$f(x) - f(y) = f'(z)(x-y)$$

$$\Rightarrow \quad |f(x) - f(y)| \leq |f'(z)|\,|x-y|$$

$$\leq \left(\sup_{[0,1]} |f'(z)| \right) |x-y| .$$

Since $|f'(z)| < 1$ & $f'(z)$ cts on $[0,1]$,

$$\gamma = \sup_{[0,1]} |f'(z)| \in [0,1).$$

If $\gamma = 0$, then $f \equiv c$ on $[0,1] \Rightarrow f(c) = c$.

If $\gamma \neq 0$, then $\gamma \in (0,1)$ & $|f(x)-f(y)| \le \gamma |x-y|$ $\forall x, y \in [0,1]$.

$\Rightarrow$ f is a contraction on the complete metric space $([0,1]$, standard$)$.

By contraction mapping principle, f has a fixed point. $\times$

§3.3 The Inverse Function Theorem

Notation: Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at a point p in an open set U of $\mathbb{R}^n$. We write

$$F = \begin{pmatrix} f^1 \\ \vdots \\ f^m \end{pmatrix} \in \mathbb{R}^m,$$

where $f^i = f^i(x^1, \cdots, x^n) : U \to \mathbb{R}$, $\forall i = 1, \cdots, m$.

Then F differentiable at $p_0 = \begin{pmatrix} x_0^1 \\ \vdots \\ x_0^n \end{pmatrix} \in U \subset \mathbb{R}^n$

$\Rightarrow$

$$F(p_0+z) - F(p_0) = DF(p_0)\,z + o(z)$$

$\forall\ z = \begin{pmatrix} z^1 \\ \vdots \\ z^n \end{pmatrix}$ sufficiently small, ($|z|$ small)

where

$$DF(p_0) = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial x^1} & \cdots & \frac{\partial f^m}{\partial x^n} \end{pmatrix}$$

or $\begin{pmatrix} f^1_1 \cdots f^1_n \\ \vdots \quad\ \vdots \\ f^m_1 \cdots f^m_n \end{pmatrix}$

i.e. $$\big(DF(p_0)z\big)^i = \sum_{j=1}^n \frac{\partial f^i}{\partial x^j}(p_0)\, z^j \quad \forall i = 1, \cdots, m.$$