eg Let $P = \{ f \in C[a,b] : f(x) = p(x)$ on $[a,b]$ for some polynomial $p(x) \}$.

Then P is not complete (in d_∞-metric):

$$h_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \in P$$

but $h_n(x) \to e^x$ uniformly (in d_∞-metric)

& $e^x \notin P$.

Def: A metric space (X, d) is said to be isometrically embedded in metric space (Y, ρ) if $\exists$ a mapping $\Phi : X \to Y$ s.t.

$$d(x,y) = \rho(\Phi(x), \Phi(y)).$$

Notes: (i) Φ is called an isometric embedding from (X, d) into (Y, ρ). And sometime called a metric preserving map.

(ii) Φ must be one-to-one and continuous.

Def: Let (X, d) and (Y, ρ) be metric spaces. We call (Y, ρ) a completion of (X, d) if (1) (Y, ρ) is complete.

(2) $\exists$ isometric embedding $\Phi : (X, d) \to (Y, \rho)$

such that the closure $\overline{\Phi(X)} = Y$.

e.g.: $(Y, \rho) = (\mathbb{R}, \text{standard})$, $X = \mathbb{Q} \subset \mathbb{R}$

$(X, d) = (\mathbb{Q}, \text{induced metric})$

Then • $(\mathbb{R}, \text{standard})$ is complete;

• $\Phi : (\mathbb{Q}, \text{induced metric}) \to (\mathbb{R}, \text{standard})$

$$q \in \mathbb{Q} \longmapsto q \in \mathbb{R}$$

• & $\overline{\mathbb{Q}} = \mathbb{R}$ ($\mathbb{Q}$ is dense in $\mathbb{R}$)

$\overline{\mathbb{Q}} = \overline{\Phi(\mathbb{Q})}$

Thm 3.2 Every metric space has a completion.

Pf (Sketch of Proof)

Let (X, d) be metric space.

Let $\mathcal{C} = \{\{x_n\} \subset X : \{x_n\} \text{ Cauchy sequence}\}$

Define equivalent relation $\sim$ on $\mathcal{C}$ by

$$\{x_n\} \sim \{y_n\} \iff d(x_n, y_n) \to 0 \text{ as } n \to \infty.$$

Let $\tilde{\mathcal{C}} = \mathcal{C}/\sim$ the quotient space.

Define $\tilde{d} = \tilde{\mathscr{C}} \times \tilde{\mathscr{C}} \to \mathbb{R}$ by the following:

For $\tilde{x}$ = equi. class $[\{x_n\}]$

$\tilde{y}$ = equi. class $[\{y_n\}]$,

$$\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n \to \infty} d(x_n, y_n)$$

Then $\tilde{d}$ is well-defined and is a metric on $\tilde{\mathscr{C}}$.

One then proves $(\tilde{\mathscr{C}}, \tilde{d})$ is complete.

$$\Phi : (X, d) \longrightarrow (\tilde{\mathscr{C}}, \tilde{d}) \text{ defined by}$$
$$x \longmapsto [\{x, x, x, \cdots\}]$$

is an isometric embedding.

And one can show that

$$\overline{\Phi(X)}^{\text{closure in } (\tilde{\mathscr{C}}, \tilde{d})} = \tilde{\mathscr{C}}.$$

\#

Def: Two metric spaces (X, d), (X', d') are called <u>isometric</u> if $\exists$ <u>bijective</u> isometric embedding from (X, d) onto (X', d').

Notes : (i) the inverse of the bijective isometric embedding is also an isometric embedding.

(ii) Two metric spaces will be regarded as the same if they are isometric.

Thm : If (Y, ρ) & (Y', ρ') are completions of a metric space (X, d). Then (Y, ρ) and (Y', ρ') are isometric.

i.e. Completion is unique up to isometry.

§3.2 The Contraction Mapping Principle

Def : (1) Let (X, d) be a metric space. A map $T: (X, d) \to (X, d)$ is called a contraction if $\exists$ constant $\gamma \in (0,1)$, such that $d(Tx, Ty) \le \gamma\, d(x,y)$, $\forall x, y \in X$.

(2) A point $x \in X$ is called a fixed point of T if $Tx = x$.

(Usually we write Tx instead of $T(x)$.)