§2.4 Sequentially Compactness and Compactness

Sequentially compactness:

Def: Let E be a subset of a metric space (X, d).
We call E sequentially compact if every sequence in E contains a convergent subsequence with limit in E.

The empty set is defined to be sequentially compact.

eg: Any closed and bounded subset in $(\mathbb{R}^n, \text{standard})$ is sequentially compact. (Standard = Euclidean metric on $\mathbb{R}^n$)

eg 2.20 Let $\mathcal{S} = \{ \text{bounded sequences } \{a_k\}_{k=1}^{\infty}, a_k \in \mathbb{R} \}$

$$= \{ \{a_k\}_{k=1}^{\infty} : a_k \in \mathbb{R}, \sup_{k \geq 1} |a_k| < +\infty \}.$$

Define $\forall\ a = \{a_k\}_{k=1}^{\infty}$ & $b = \{b_k\}_{k=1}^{\infty} \in \mathcal{S}$,

$$d(a,b) = \sup_{k \geq 1} |a_k - b_k| \quad \left(\text{"sup" exists since } a, b \text{ are bdd seq.s} \right)$$

Then d is a metric on $\mathcal{S}$ (Ex!)

Let $E = \{ a \in \mathcal{S} : 0 \leq a_k \leq 1, \ \forall k \geq 1 \}$

Def: A set E in a metric is said to be <u>bounded</u> if it is contained in some metric ball (of finite radius): i.e. $\exists\, B_r(a)$ s.t.

$$E \subset B_r(a) \quad (\, a = \text{pt. in the metric sp.}, \ r \in \mathbb{R}^+ .)$$

Then according to this definition, E is <u>bounded</u> in $(\mathcal{S}, d)$:

$$E \subset B_{1+\varepsilon}(0) \ , \quad \forall\, \varepsilon > 0$$

$$\text{and} \quad 0 = (0, 0, 0, \cdots\cdots) \in \mathcal{S}.$$

Similar to eg 2.19 of the previous section, E is <u>closed</u>.

$\therefore$ E is a <u>closed and bounded</u> subset of $\mathcal{S}$.

However, let $\{a^{(n)}\}_{n=1}^{\infty}$ be a seq. of E defined by

$$a_k^{(n)} = \begin{cases} 0 & \text{if } k \neq n \\ 1 & \text{if } k = n \end{cases} .$$

i.e. $a^{(n)} = (0, 0, \cdots, 0, 1, 0, \cdots, 0, \cdots\cdots) \in E \subset \mathcal{S}$

$\uparrow$ n-th place

Then $\quad d(a^{(m)}, a^{(n)}) = 1 \quad , \quad \text{if } m \neq n.$

Hence, there is no convergent subsequence

(otherwise, $d(a^{(n_i)}, a^{(n_j)}) < \varepsilon < 1$, for some i, j)

$\therefore$ E is <u>not</u> sequentially compact.

($\therefore$ Closed & bdd $\not\Longrightarrow$ compactness

for general metric space.).

<u>eg 2.21</u> ($X = C[0,1]$, d_∞)

$$E = \{ f \in C[0,1] : 0 \leq f(x) \leq 1 \}$$

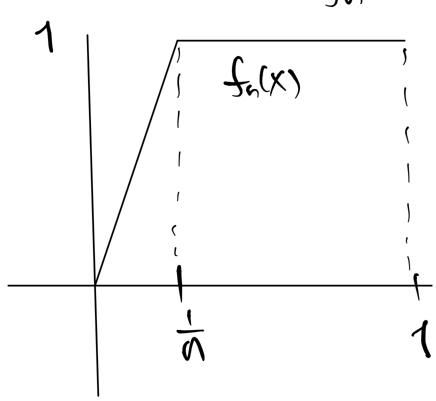
is <u>closed</u> (eg 2.19) and

<u>bounded</u> as $E \subset B^{\infty}_{1+\varepsilon}(0)$, ($0$ is the zero function)

<u>Claim</u>: E is not sequentially compact.

<u>Pf</u>: Consider

$$f_n(x) = \begin{cases} nx, & x \in [0, \frac{1}{n}] \\ 1, & x \in [\frac{1}{n}, 1] \end{cases}$$



$f_n \in E$, $\forall n$

If E is sequentially compact, then $\exists$ a subseq.

$\{f_{n_j}\} \subset \{f_n\} \subset E$ s.t.

$f_{n_j} \to f$ in $(\mathcal{B}, d_\infty)$, for some $f \in E$.

i.e. $\|f_{n_j} - f\|_\infty \to 0$ as $n_j \to \infty$

$\Rightarrow$ f_{n_j} converges to f uniformly.

$\Rightarrow$ $f(x) = \begin{cases} 1, & \forall x \neq 0 \\ 0, & x = 0 \end{cases}$ (discontinuous at $x=0$)

This is a contradiction.

$\therefore$ E is not sequentially compact. #