

MATH3060

HW4

Due Date: Oct 12, 2016

1. Draw the unit metric balls $B_r(0)$, $B'_r(0)$ and $B_r^\infty(0)$ (with $r=1$) for metrics d_2 , d_1 and d_∞ on \mathbb{R}^2 respectively.

2. Let (X, d) be a metric space and define

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

Show that

(a) ρ is a metric on X .

(b) A sequence converges in d if and only if it converges in ρ .

(c) If ρ is equivalent to d , then $\exists C > 0$ such $d(x, y) \leq C$, $\forall x, y \in X$.

3. Give an example of 2 inequivalent metrics ρ and d which have the same concept of convergence, i.e. convergence in $d \Leftrightarrow$ convergence in ρ .

4. Show that d_2 is stronger than d_1 on $C[a, b]$ but they are not equivalent.

5. A "functional" is a real-valued function defined on a space of functions. Show that the following functionals are continuous with respect to the given metric. (\mathbb{R} always equipped with the standard metric $d(x,y) = |x-y|$, $\forall x,y \in \mathbb{R}$)

(a) $\Phi: (C[a,b], d_1) \rightarrow \mathbb{R}$ given by

$$\Phi(f) = \int_a^b \sqrt{1+f^2(x)} dx$$

(b) $\Phi: (C[a,b], d_\infty) \rightarrow \mathbb{R}$ with same

Φ as in (a).

(c) $\Psi: (C[-1,1], d_\infty) \rightarrow \mathbb{R}$ given by

$$\Psi(f) = f(0).$$

6. Show that $\Psi: (C[-1,1], d_1) \rightarrow \mathbb{R}$ given by

$\Psi(f) = f(0)$ is not continuous.