Proof of Thm 1.5

Let f be Lip cts at a point $x_0 \in [-\pi, \pi]$.

Step 1: $(S_n f)(x_0) = a_0 + \sum_{k=1}^{n} (a_k \cos kx_0 + b_k \sin kx_0)$

$$= \int_{-\pi}^{\pi} D_n(z) f(x_0 + z) dz$$

where
$$D_n(z) = \begin{cases} \dfrac{\sin (n+\frac{1}{2}) z}{2\pi \sin \frac{1}{2} z}, & \text{if } z \neq 0 \\[2ex] \dfrac{2n+1}{2\pi}, & \text{if } z = 0 \end{cases}$$

is called the Dirichlet kernel.

Pf: $(S_n f)(x_0) = a_0 + \sum_{k=1}^{n} (a_k \cos kx_0 + b_k \sin kx_0)$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) dy + \sum_{k=1}^{n} \left[\left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos ky \, dy \right) \cos kx_0 + \left(\frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin ky \, dy \right) \sin kx_0 \right]$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2} + \sum_{k=1}^{n} (\cos ky \cos kx_0 + \sin ky \sin kx_0) \right] f(y) dy$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2} + \sum_{k=1}^{n} \cos k(y - x_0) \right] f(y) dy$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2}+\sum_{k=1}^{\infty}\cos kz\right] f(z+x_0)\,dz \qquad \begin{pmatrix} z=y-x_0 \end{pmatrix}$$

& 2π-periodic

Since $\displaystyle \frac{1}{2}+\sum_{k=1}^{n}\cos kz = \frac{\sin(n+\frac{1}{2})z}{2\sin\frac{1}{2}z}$ for $z\neq 0$,

(Ex: Calculate $e^{-in\theta}+\cdots+1+\cdots+e^{in\theta}$ using $1+z+\cdots+z^k = \dfrac{z^{k+1}-1}{z-1}$.)

$$(S_nf)(x_0) = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin(n+\frac{1}{2})z}{2\sin\frac{1}{2}z} f(x_0+z)\,dz$$

$$= \int_{-\pi}^{\pi} D_n(z) f(x_0+z)\,dz \qquad \#$$

Step 2 (Properties of $D_n(z)$)

(1) $\int_{-\pi}^{\pi} D_n(z)\,dz = 1$

(2) $D_n(z)$ is even, cts, 2π-periodic on $[-\pi,\pi]$,
& $D_n\left(\frac{2k\pi}{2n+1}\right)=0$ for $k=-n,\cdots,0,\cdots,n$

(3) $\max\limits_{[-\pi,\pi]} D_n(z) = D_n(0) = \frac{2n+1}{2\pi}$

(4) $\forall\delta>0$, $\int_0^{\delta}|D_n(z)|\,dz \to +\infty$ as $n\to+\infty$.

Pf: (1) Easy: by integrating $\int_{-\pi}^{\pi}\left(\frac{1}{2}+\sum_{k=1}^{n}\cos kz\right)dz$.

(2) & (3) are easy exercise.

(4) Reading exercise (KS Chou's note).

<u>Step 3</u> Splitting $(S_n f)(x_0) - f(x_0) = \mathrm{I} + \mathrm{II}$ into integrals concentrated in $[-\delta, \delta]$ & (essential) outside $[-\delta, \delta]$.

By (1) in Step 2,

$$f(x_0) = \int_{-\pi}^{\pi} D_n(z) f(x_0)\, dz$$

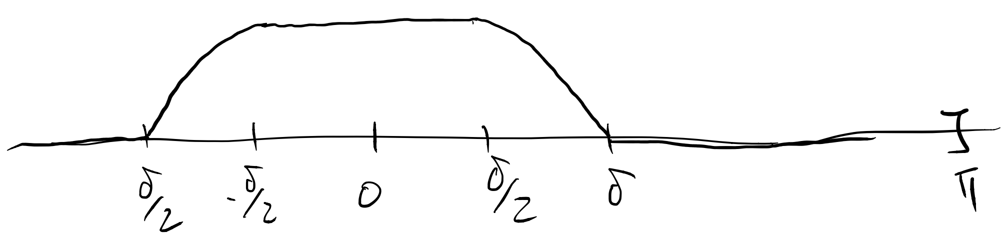
$$\therefore \quad (S_n f)(x_0) - f(x_0) = \int_{-\pi}^{\pi} D_n(z)\left[f(x_0+z) - f(x_0)\right] dz$$

Let Φ_δ be a "cut-off" function s.t.

(i) Φ_δ is cts & $0 \leq \Phi_\delta \leq 1$

(ii) $\Phi_\delta(t) \equiv 1$ for $|t| \leq \frac{\delta}{2}$

(iii) $\Phi_\delta(t) \equiv 0$ for $|t| \geq \delta$.



Then $(S_n f)(x_0) - f(x_0)$

$$= \int_{-\pi}^{\pi} D_n(z)\left[f(x_0+z) - f(x_0)\right] dz$$

$$= \int_{-\pi}^{\pi} \Phi_\delta(z) D_n(z)\left[f(x_0+z) - f(x_0)\right] dz$$

$$+\int_{-\pi}^{\pi}\left(1-\Phi_\delta(z)\right)D_n(z)\left[f(x_0+z)-f(x_0)\right]dz$$

$$= \mathrm{I} + \mathrm{II}$$

where

$$\mathrm{I} = \int_{-\pi}^{\pi} \Phi_\delta(z) D_n(z)\left[f(x_0+z)-f(x_0)\right]dz$$

$$= \int_{-\delta}^{\delta} \Phi_\delta(z) D_n(z)\left[f(x_0+z)-f(x_0)\right]dz$$

$$\mathrm{II} = \int_{-\pi}^{\pi} (1-\Phi_\delta(z))\, D_n(z)\left[f(x_0+z)-f(x_0)\right]dz$$

$$= \left(\int_{-\pi}^{-\delta/2} + \int_{\delta/2}^{\pi}\right)(1-\Phi_\delta(z))D_n(z)\left[f(x_0+z)-f(x_0)\right]dz$$

<u>Step 4</u>: $\exists\, L>0$ and $\delta_2>0$ such that

$$|\mathrm{I}| \le \frac{4\delta L}{\pi}, \quad \forall\ 0<\delta<\delta_2.$$

<u>Pf</u>: By Lip cts at x_0, $\exists\, L>0$ & $\delta_0>0$ s.t.

$$|f(x_0+z)-f(x_0)| \le L|z|, \quad \forall\ |z|<\delta_0$$

Since $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$, $\exists\, \delta_1>0$ s.t.

$$\left|\frac{\sin\frac{z}{2}}{\frac{z}{2}}\right| > \frac{1}{2}, \quad \forall\ |z|<\delta_1$$

Therefore, for $\delta_2 = \min\{\delta_0, \delta_1\} > 0$,

$$\frac{|f(x_0+z)-f(x_0)|}{|\sin\frac{z}{2}|} \leq \frac{L|z|}{\frac{1}{2}|\frac{z}{2}|} = 4L, \quad \forall |z|<\delta_2$$

Hence $\forall\ 0<\delta<\delta_2$, we have

$$|\mathrm{I}| \leq \int_{-\delta}^{\delta} |\Phi_\delta(z)|\,|D_n(z)|\,|f(x_0+z)-f(x_0)|\,dz$$

$$= \int_{-\delta}^{\delta} |\Phi_\delta(z)| \frac{|\sin(n+\frac{1}{2})z|}{2\pi\,|\sin\frac{z}{2}|} |f(x_0+z)-f(x_0)|\,dz$$

$$\leq \int_{-\delta}^{\delta} 1\cdot\frac{1}{2\pi}\cdot 4L\,dz = \frac{4\delta L}{\pi} \quad \#$$

Step 5: $\forall \varepsilon>0$, $\exists \delta>0$ & $n_0>0$ s.t.

$$\frac{4\delta L}{\pi} < \varepsilon/2 \quad \text{and} \quad |\mathrm{II}| < \varepsilon/2, \quad \forall n\geq n_0$$

Pf: $\forall \varepsilon>0$, we take

$$\delta = \min\left\{\frac{\varepsilon\pi}{8L}, \delta_2\right\} > 0 \quad \left(\begin{array}{c} L \ \&\ \delta_2 \text{ as} \\ \text{in step 4}\end{array}\right)$$

Then $\frac{4\delta L}{\pi} < \varepsilon/2$,

and for this fixed $\delta>0$,

$$\mathrm{II} = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1-\Phi_\delta(z))\frac{\sin(n+\frac{1}{2})z}{\sin\frac{z}{2}}[f(x_0+z)-f(x_0)]\,dz$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-\Phi_\delta(z))\,[f(x_0+z)-f(x_0)]}{\sin\frac{z}{2}} \left[\sin nz \cos\frac{z}{2} + \cos nz \sin\frac{z}{2}\right] dz$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \left[\frac{(1-\Phi_\delta(z))\,[f(x_0+z)-f(x_0)]}{2\sin\frac{z}{2}} \cos\frac{z}{2}\right] \sin nz\, dz$$

$$+ \frac{1}{\pi}\int_{-\pi}^{\pi} \left[\frac{(1-\Phi_\delta(z))}{2}[f(x_0+z)-f(x_0)]\right] \cos nz\, dz$$

$$= b_n(F_1) + a_n(F_2)$$

where $$F_1(z) = \frac{[1-\Phi_\delta(z)]\,[f(x_0+z)-f(x_0)]\cos\frac{z}{2}}{2\sin\frac{z}{2}}$$

$$F_2(z) = \frac{(1-\Phi_\delta(z))}{2}[f(x_0+z)-f(x_0)]$$

(cont'd next time)