§ 63 Proof of Taylor's Theorem

Pf: Special case $z_0 = 0$.

ie f analytic in $|z| < R_0$.

Then $\forall\ r_0 > 0$ s.t. $0 < r_0 < R_0$,

the function f is analytic inside and on

the circle $C_0 = \{ |z| = r_0 \} \subset \{ |z| < R_0 \}$.

Cauchy integral formula $\Longrightarrow$

$$f(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(s)\,ds}{s - z}, \quad \forall\ |z| < r_0$$

$$= \frac{1}{2\pi i} \int_{C_0} f(s) \left[\frac{1}{s(1 - \frac{z}{s})} \right] ds, \quad \forall |z| < r_0$$

$$= \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s} \left[1 + \frac{z}{s} + \cdots + \left(\frac{z}{s}\right)^{N-1} + \frac{\left(\frac{z}{s}\right)^N}{1 - \left(\frac{z}{s}\right)} \right] ds$$

$$\left(\text{since} \quad \left|\frac{z}{s}\right| = \frac{|z|}{r_0} < 1 \right)$$

$$= \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s} \left[\sum_{n=0}^{N-1} \left(\frac{z}{s}\right)^n + \frac{s}{s-z} \left(\frac{z}{s}\right)^N \right] ds$$

$$= \sum_{n=0}^{N-1} \left(\frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s^{n+1}} ds \right) z^n + \frac{1}{2\pi i} \int_{C_0} \frac{f(s)}{s-z} \left(\frac{z}{s}\right)^N ds$$

extended Cauchy Integral Formula $= \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} z^n + \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s-z}\left(\frac{z}{s}\right)^N ds$

To estimate $\rho_N(z) = \frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s-z}\left(\frac{z}{s}\right)^N ds$,

we denote $M_0 = \sup_{|s|=r_0} |f(s)|$.

Then $|s-z| \geq \big| |s| - |z| \big| = r_0 - r$ where $r = |z| < r_0$

$$\Rightarrow \left|\rho_N(z)\right| \leq \left|\frac{1}{2\pi i}\int_{C_0} \frac{f(s)}{s-z}\left(\frac{z}{s}\right)^N ds\right|$$

$$\leq \frac{1}{2\pi} \frac{M_0}{r_0 - r}\left(\frac{r}{r_0}\right)^N \cdot 2\pi r_0$$

$$= \left(\frac{M_0 r_0}{r_0 - r}\right)\left(\frac{r}{r_0}\right)^N \longrightarrow 0 \text{ as } N \to \infty$$

(since $\frac{r}{r_0} < 1$)

$$\therefore \quad f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n .$$

General z_0:

(Ex: Consider $g(z) = f(z + z_0)$!)

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§64 Examples

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1+z+z^2+\cdots \qquad (|z|<1)$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1+\frac{z}{1!}+\frac{z^2}{2!}+\cdots \qquad (|z|<\infty)$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} = z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots \qquad (|z|<\infty)$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots \qquad (|z|<\infty)$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots \qquad (|z|<\infty)$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots \qquad (|z|<\infty)$$

(Need to remember all the above 6 expansions!)

eg 1: Let $f(z) = \frac{1}{1-z}$

(i) Check that $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n = 1+z+z^2+\cdots \quad (|z|<1)$

(ii) Note that $|z|<1$, then $|-z|<1$

$$\therefore \frac{1}{1+z} = \frac{1}{1-(-z)} = 1+(-z)+(-z)^2+\cdots \qquad (|z|<1)$$

$$= 1-z+z^2-z^3+\cdots \qquad (|z|<1)$$

(iii) Let $\zeta = 1 - z$, then $|\zeta - 1| = |z| < 1$.

$$\frac{1}{\zeta} = \frac{1}{1-z} = 1 + z + z^2 + \cdots + z^n + \cdots \qquad (|z|<1)$$

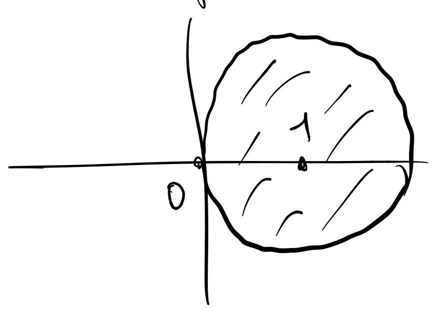
$$= 1 + (1-\zeta) + (1-\zeta)^2 + \cdots + (1-\zeta)^n + \cdots$$

$$= 1 - (\zeta - 1) + (\zeta-1)^2 + \cdots + (-1)^n(\zeta-1)^n + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n (\zeta - 1)^n \qquad (|\zeta - 1| < 1)$$

Replace ζ by z $\quad \frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n \quad (|z-1|<1)$

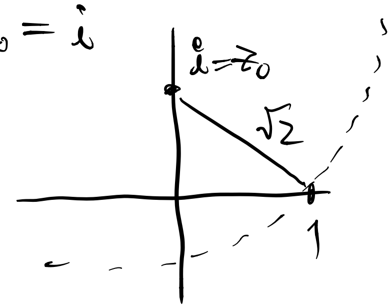
This is the Taylor series expansion for $\frac{1}{z}$ about $z_0 = 1$:



(iv) $f(z) = \frac{1}{1-z}$ is analytic at $z_0 = i$

In fact, f is analytic in

$|z - i| < \sqrt{2}$.



$$f(z) = \frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i} \frac{1}{1 - \left(\frac{z-i}{1-i}\right)}$$

Note that $\left|\frac{z-i}{1-i}\right| = \frac{|z-i|}{\sqrt{2}} < 1$,

$$f(z) = \frac{1}{1-i}\left[1 + \left(\frac{z-i}{1-i}\right) + \left(\frac{z-i}{1-i}\right)^2 + \cdots + \left(\frac{z-i}{1-i}\right)^n + \cdots\right]$$

$$= \frac{1}{1-i}\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{(1-i)^{n+1}}(z-i)^n \quad \left(|z-i|<\sqrt{2}\right)$$

is the required Taylor expansion.

(Ex: Direct check $\frac{1}{(1-i)^{n+1}} = \frac{f^{(n)}(i)}{n!}$.)

<u>eg2</u> (Easy) $f(z) = z^3 e^{2z}$

(Ex!) $$z^3 e^{2z} = z^3 \sum_{n=0}^{\infty}\frac{(2z)^n}{n!}$$

(check) $$= \sum_{k=3}^{\infty}\frac{2^{k-3}}{(k-3)!} z^k \quad (|z|<\infty)$$

(change of index $k = n+3$)

<u>eg3</u> $$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$= \frac{1}{2i}\left[\sum_{n=0}^{\infty}\frac{(iz)^n}{n!} - \sum_{n=0}^{\infty}\frac{(-iz)^n}{n!}\right]$$

$$= \frac{1}{2i} \sum_{n=0}^{\infty} \left[i^n - (-i)^n \right] \frac{z^n}{n!}$$

$$= \frac{1}{2i} \sum_{n=0}^{\infty} \left[1 - (-1)^n \right] (i)^n \frac{z^n}{n!}$$

$$\left(= \frac{1}{2i} \sum_{\text{odd}} 2\, (i)^n \frac{z^n}{n!} \right)$$

$$= \frac{1}{i} \sum_{k=0}^{\infty} (i)^{2k+1} \frac{z^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} (i)^{2k} \frac{z^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \quad (|z| < \infty)$$

egs 4, 5, 6 : Reading exercise.

Note for eg 6 : From $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$ $(|z|<\infty)$, we have

$$\cosh z = \cosh(z - 2\pi i) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} (z - 2\pi i)^{2n}$$

$|z - 2\pi i| < \infty$
$\left(\overset{\shortparallel}{|z|} < \infty \right)$

is the Taylor series expansion for $\cosh z$ about $z_0 = 2\pi i$!