Thm 2 Let f be continuous on a domain D. If

$$\int_C f(z)\,dz = 0 \quad \text{for every closed contour } C \text{ in } D,$$

then f is analytic throughout D.

Pf: If $\int_C f(z)\,dz = 0$ $\forall$ closed contour C

then f has an antiderivative F in D.

i.e. $F'(z) = f(z)$ $\forall z \in D$.

$\Rightarrow$ F is analytic in D.

$\therefore$ Thm 1, $f = F'$ is analytic $\forall z \in D$ ⁂

Thm 3 Suppose that a function f is analytic inside and on a positively oriented circle C_R, centered at z_0 and with radius R. If M_R denotes the maximum value of $|f(z)|$ on C_R,

then
$$|f^{(n)}(z_0)| \le \frac{n!\,M_R}{R^n} \quad (\forall n = 1, 2, 3, \cdots)$$

(Cauchy Inequality)

Pf: By the extended Cauchy-Goursat Thm,

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_R} \frac{f(z)\,dz}{(z-z_0)^{n+1}}, \quad n = 1, 2, \cdots$$

Note that on C_R, $|z-z_0|=R$ & length of C_R is $2\pi R$,

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \frac{M_R}{(R)^{n+1}} \cdot 2\pi R = \frac{n!\, M_R}{R^n}$$

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§58 Liouville's Theorem and the Fundamental Theorem of Algebra

Thm 1 (Liouville's Thm) If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

Pf: Let M be the bound of f, i.e. $|f(z)| \leq M$, $\forall z \in \mathbb{C}$

Then $\forall z_0 \in \mathbb{C}$ and any $R>0$,

f entire $\Rightarrow$ f analytic inside & on $C_R = \{|z-z_0|=R\}$.

$$\Rightarrow |f'(z_0)| \leq \frac{M_R}{R} \quad \text{(by Cauchy Inequality)}$$
$$\leq \frac{M}{R}$$

Since $R>0$ is arbitrary, we have $f'(z_0)=0$ by letting $R \to \infty$. $\therefore f' \equiv 0$ on $\mathbb{C}$

$\Rightarrow f \equiv \text{constant}$ on $\mathbb{C}$. ※

<u>Thm 2</u> (<u>Fundamental Theorem of Algebra</u>)

Any polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, $(a_n \neq 0)$ of degree n $(n \geq 1)$ has at least one zero.

(i.e. $\exists\, z_0 \in \mathbb{C}$ s.t. $P(z_0) = 0$.)

<u>Pf</u>: Suppose not. Then $P(z) \neq 0$, $\forall z \in \mathbb{C}$

$\Rightarrow f(z) = \dfrac{1}{P(z)}$ is an entire function.

Now for $z \neq 0$,

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$
$$= z^n \left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right).$$

For $k = 0, 1, \cdots, n-1$

$$\left| \frac{a_k}{z^{n-k}} \right| \leq \frac{|a_k|}{|z|^{n-k}} < \frac{|a_n|}{2n} \quad \text{if } |z| > \sqrt[n-k]{\frac{2n|a_k|}{|a_n|}}$$

Therefore, for $R = \max\limits_{k=0,1,\cdots,n-1} \left(\sqrt[n-k]{\dfrac{2n|a_k|}{|a_n|}} \right) > 0$,

we have, for $|z| > R$, $\left| \dfrac{a_k}{z^{n-k}} \right| < \dfrac{|a_n|}{2n}$, $\forall k = 0, 1, \cdots, n-1$

$$\Rightarrow \left| \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| \leq n \cdot \frac{|a_n|}{2n} = \frac{|a_n|}{2}.$$

$$\therefore \quad |P(z)| = |z^n| \left| a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right|$$

$$\geq |z|^n \left| |a_n| - \left| \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right| \right|$$

$$\geq |z|^n \left(|a_n| - \frac{|a_n|}{2} \right)$$

$$= \frac{|a_n|}{2} |z|^n > \frac{|a_n| R^n}{2}$$

$$\therefore \quad |f(z)| = \left| \frac{1}{P(z)} \right| < \frac{2}{|a_n| R^n}, \quad \forall |z| > R.$$

Note that $P(z) \neq 0$, $\forall z \in \mathbb{C} \Rightarrow f(z) = \frac{1}{P(z)}$ is cts on $\overline{B_R(0)}$ (closed & bounded set in $\mathbb{C}$)

$\Rightarrow \exists M_1 > 0$ s.t. $|f(z)| \leq M_1$, $\forall z \in \overline{B_R(0)}$.

All together $|f(z)| \leq \max\{M_1, \frac{2}{|a_n| R^n}\}$, $\forall z \in \mathbb{C}$

$\therefore \quad f(z) = \frac{1}{P(z)}$ is bounded.

Then Liouville's Thm $\Rightarrow f(z) = \frac{1}{P(z)} = \text{const}.$

$\Rightarrow n = 0$ which is a contraction (as $n \geq 1$)

$\therefore \quad P(z)$ has a zero. $\#$

Note: By fundamental thm of algebra, we immediately have $P(z) = a_n(z-z_1)\cdots(z-z_n)$

where $z_1, \cdots, z_n$ are zeroes (may not distinct) of P.