Note: eg1 shows that

$$\int_{C_1} f(z)dz \neq \int_{C_2} f(z)dz$$

even C_1 and C_2 have the same initial and end points 1 & -1.

$\therefore$ Contour integrals are path dependent in general.

$\Rightarrow$ $\int_{z_1}^{z_2} f(z)dz$ may not be defined!

However, we will write $\int_{z_1}^{z_2} f(z)dz$ when it is independent of the contour joining z_1 to z_2.

eg2: Let $C: z=z(t)$, $a \leq t \leq b$, $z_1 = z(a)$ & $z_2 = z(b)$.

Then

$$\int_C z\,dz = \int_a^b z(t)\,z'(t)\,dt$$

$$= \frac{1}{2}\int_a^b \frac{d}{dt}[z(t)]^2$$

$$= \frac{1}{2}[z(b)^2 - z(a)^2]$$

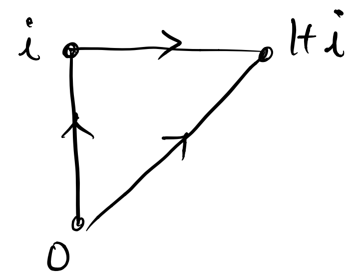
$$= \frac{1}{2}(z_2^2 - z_1^2)$$ depends only on the initial & end points, not the path.

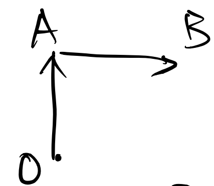
$\therefore$ In this case, we write $\int_{z_1}^{z_2} z\,dz = \frac{1}{2}(z_2^2 - z_1^2)$. #

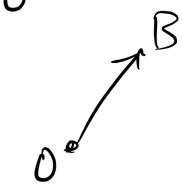
eg3: Let $f(z) = y - x - i3x^2$, $z = x+iy$.

(Note: $u = y - x$, $v = -3x^2$

$\Rightarrow \begin{cases} u_x = -1 & v_x = -6x \\ u_y = 1 & v_y = 0 \end{cases}$, not analytic)

Let 

C_1: 

C_2: 

$$\int_{C_1} f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz$$

$$= \int_0^1 f(iy)d(iy) + \int_0^1 f(x+i)d(x+i)$$

$$= \int_0^1 y\,i\,dy + \int_0^1 (1 - x - i3x^2)\,dx$$

$$= \int_0^1 (1-x)dx + i\left(\int_0^1 y\,dy - \int_0^1 3x^2 dx\right)$$

$$= \frac{1-i}{2} \quad \text{(check!)}$$

$$\int_{C_2} f(z)dz = \int_0^1 f(x+ix)\,d(x+ix)$$

$$= \int_0^1 (x - x - i3x^2)(1+i)\,dx$$

$$= 1 - i \quad (\text{Check!})$$

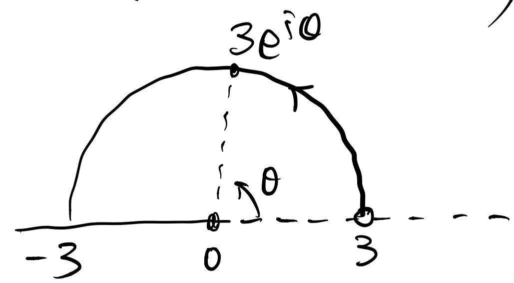
$$\neq \frac{1-i}{2} = \int_{C_1} f(z)\,dz$$

※

§46 Examples Involving Branch Cuts

eg1 : Let C : $z = 3e^{i\theta}$, $0 \le \theta \le \pi$ (semi circular arc)

$f(z) = z^{1/2}$



Suppose we consider the following branch of $z^{1/2}$:

$$f(z) = z^{1/2} = \exp\left(\tfrac{1}{2}\log z\right), \quad |z| > 0,\ 0 < \arg z < 2\pi$$

Then the initial point $z(0) = 3$ (of the arc C) doesn't belongs to this branch. However, for this branch,

$$f(z(\theta))\,z'(\theta) = \sqrt{3}\,e^{\frac{i\theta}{2}} \cdot 3ie^{i\theta} = 3\sqrt{3}\,i\,e^{\frac{i3\theta}{2}}$$

$$0 < \theta \le \pi$$

As $f(z(\theta))z'(\theta) \to 3\sqrt{3}\,i$ as $\theta \to 0$,

$f(z(\theta))z'(\theta)$ is piecewise smooth on $[0, \pi]$.

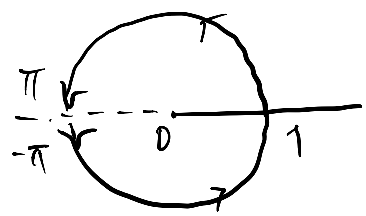
$\therefore \int_C z^{1/2} dz$ exists and

$$= \int_0^{\pi} 3\sqrt{3}\, i e^{i\frac{3\theta}{2}} d\theta = 2\sqrt{3}\left[e^{i\frac{3\theta}{2}}\right]_0^{\pi}$$

$$= -2\sqrt{3}(1+i)$$

eg: Evaluate $\int_C z^{-1+i} dz$ in principal branch along the unit circle C.

Soln: Principal branch of



$$z^{-1+i} = \exp\left[(-1+i)\operatorname{Log} z\right], \quad -\pi < \operatorname{Arg} z < \pi$$

$(|z|>0)$

$$= \exp\left[(-1+i)(\ln|z| + i\operatorname{Arg} z)\right].$$

The unit circle can be represented by

$$C: z = e^{i\theta}, \ -\pi \leq \theta \leq \pi.$$

(Then $\theta = \operatorname{Arg} z$ except only at the end points)

$$\therefore \int_C z^{-1+i} dz = \int_{-\pi}^{\pi} e^{(-1+i)i\theta} \cdot ie^{i\theta} d\theta$$

$$= i\int_{-\pi}^{\pi} e^{-\theta} d\theta = i\left[-e^{-\theta}\right]_{-\pi}^{\pi} = i(-e^{-\pi} + e^{\pi})$$

$$= 2i \sinh \pi$$

※

§47 Upper Bounds for Moduli of Contour Integrals

Lemma: If $w(t)$ is a piecewise continuous cpx-valued function defined on $a \le t \le b$, then

$$\left| \int_a^b w(t)\,dt \right| \le \int_a^b |w(t)|\,dt$$

Pf: If $\int_a^b w(t)\,dt = 0$, then we are done.

If $\int_a^b w(t)\,dt \neq 0$, then it can be written as

$$\int_a^b w(t)\,dt = r_0 e^{i\theta_0},$$

where $r_0 = \left| \int_a^b w(t)\,dt \right| > 0$ & $\theta_0 \in \mathbb{R}$.

Then

$$\begin{aligned} r_0 &= e^{-i\theta_0} \int_a^b w(t)\,dt \\ &= \int_a^b e^{-i\theta_0} w(t)\,dt \\ &= \mathrm{Re}\left[\int_a^b e^{-i\theta_0} w(t)\,dt \right] \quad (\text{since } r_0 \in \mathbb{R}) \\ &= \int_a^b \mathrm{Re}\left[e^{-i\theta_0} w(t) \right] dt \\ &\le \int_a^b |e^{-i\theta_0} w(t)|\,dt \\ &= \int_a^b |w(t)|\,dt \quad ※ \end{aligned}$$

Thm: Let C be a contour of length L, and $f(z)$ be a piecewise continuous function on C.

Suppose $M > 0$ is constant s.t.

$$|f(z)| \le M, \quad \forall z \in C.$$

Then
$$\left| \int_C f(z)\,dz \right| \le ML.$$

Pf: Parametrize C by $z = z(t)$, $a \le t \le b$.

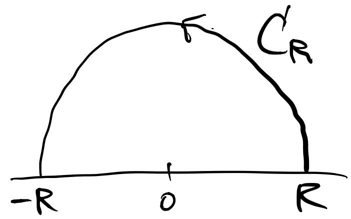
Then $\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\,dt$

By lemma $\Rightarrow$
$$\left| \int_C f(z)\,dz \right| \le \int_a^b |f(z(t))|\,|z'(t)|\,dt$$
$$\le M \int_a^b |z'(t)|\,dt = ML. \quad \#$$

eg 1 (Reading Ex!)

eg 2 Let C_R = semicircle $z = Re^{i\theta}$, $0 \le \theta \le \pi$

for $R > 3$.



Show that
$$\lim_{R\to\infty} \int_{C_R} \frac{(z+1)\,dz}{(z^2+4)(z^2+9)} = 0.$$

Pf: For $R > 3$, we have on C_R that
$$\begin{cases} |z+1| \le |z|+1 = R+1 \\ |z^2+4| \ge |z|^2 - 4 = R^2 - 4 \ (>0) \\ |z^2+9| \ge |z|^2 - 9 = R^2 - 9 \ (>0) \end{cases}$$

$$\Rightarrow \left|\frac{z+1}{(z^2+4)(z^2+9)}\right| \le \frac{R+1}{(R^2-4)(R^2-9)}, \quad \forall z \in C_R.$$

Note that length of $C_R = \pi R$, we have

$$\left|\int_{C_R}\frac{(z+1)\,dz}{(z^2+4)(z^2+9)}\right| \le \frac{R+1}{(R^2-4)(R^2-9)}\cdot \pi R$$

$$\longrightarrow 0 \text{ as } R\to+\infty.$$

#

§ 48 Antiderivatives

Def: Let $f(z)$ be a cpx-valued cts. function in a domain D. Then the antiderivative of $f(z)$ on D is a function $F(z)$ such that $F'(z) = f(z)$, $\forall z \in D$.

Notes: (i) An antiderivative is an analytic function.

(ii) An antiderivative of a given function is unique up to an additive constant:

i.e. if F & G are antiderivatives of f, then $F-G$ is a constant function.

(since D is connected!)

Thm: Suppose that a function $f(z)$ is cts. in a domain D. Then the following statements are equivalent.

(a) $f(z)$ has an antiderivative $F(z)$ throughout D.

(b) $\forall$ contours C_1 and C_2 (lying entirely in D) with the same initial & end points,

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz .$$

(c) $\forall$ closed contour C (lying entirely in D),

$$\int_C f(z)dz = 0 .$$

If any of the above statements true, then the integral in (b)

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)dz = F(z)\Big|_{z_1}^{z_2} = F(z_2) - F(z_1),$$

where F is the antiderivative given in (a).

So we denote, in this case,

$$\boxed{\int_{z_1}^{z_2} f(z)dz = F(z)\Big|_{z_1}^{z_2} = F(z_2) - F(z_1)} .$$

eg1: let $f(z) = e^{\pi z}$ & $F(z) = \frac{1}{\pi} e^{\pi z}$ on $\mathbb{C}$.

Then $\forall z \in \mathbb{C}$, $F'(z) = f(z)$.

$\Rightarrow$ $f(z)$ has antiderivative $F(z)$ on the whole $\mathbb{C}$.

$\Rightarrow$ for any contour C with initial point z_1 & end point z_2,

$$\left(\int_{z_1}^{z_2} e^{\pi z} dz \mathbin{/\!\!/} \right) \int_C f(z)dz = \int_C e^{\pi z} dz$$

$$= \frac{1}{\pi} e^{\pi z} \Big|_{z_1}^{z_2} = \frac{1}{\pi}\left(e^{\pi z_2} - e^{\pi z_1}\right).$$

eg2 : $f(z)=\frac{1}{z^2}$ cts. on $\mathbb{C}\setminus\{0\}$

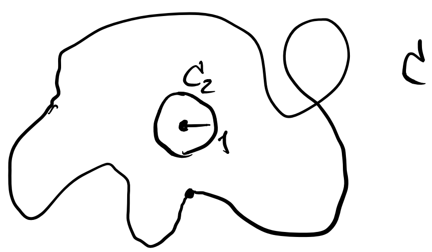
Note that $F(z)=-\frac{1}{z}$ is analytic on $\mathbb{C}\setminus\{0\}$ and

$$F'(z)=f(z),\ \forall\, z\in\mathbb{C}\setminus\{0\}.$$

Then by part (c) of the Thm :

$$\int_C \frac{1}{z^2}\,dz=0 \quad \forall \text{ closed contour in } \mathbb{C}\setminus\{0\}.$$

$$\left(\int_{\substack{\text{unit circle}\\ C_2}} \frac{1}{z^2}\,dz=0. \right)$$



eg3 : However, we have seen $\displaystyle\int_{\text{unit circle}} \frac{dz}{z}=2\pi i\neq 0$

$$\left(\int_{\substack{\text{unit}\\ \text{circle}}} \frac{dz}{z}=\int_{C_1}\frac{dz}{z}+\int_{-C_2}\frac{dz}{z}=\pi i+\pi i=2\pi i \right)$$

in eg 1 of §45

The <u>issue</u> is $f(z)=\frac{1}{z}$ in $\mathbb{C}\setminus\{0\}$ has <u>NO</u>

antiderivative $F(z)$ throughout $\mathbb{C}\setminus\{0\}$.

We can <u>at most</u> find an antiderivative in

$$\mathbb{C}\setminus\{\text{a ray}\}=\mathbb{C}\setminus\{\text{branch cut}\},$$

namely a branch of $\log z$!