§ 22 Examples

eg1: $f(z) = z^2$ is differentiable & $f'(z) = 2z$

$$= (x^2-y^2) + i(2xy)$$

$\therefore \begin{cases} u = x^2 - y^2 \\ v = 2xy \end{cases}$

$$\begin{cases} u_x = 2x & u_y = -2y \\ v_x = 2y & v_y = 2x \end{cases}$$

satisfy C-R eqt: $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$

& $u_x + i v_x = 2x + i2y = 2z = f'(z)$.

(eg2: later in next section)

eg3 $f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

For $z \neq 0$ $\quad f(z) = \dfrac{x^3 - 3xy^2}{x^2+y^2} + i \dfrac{(-3x^2y + y^3)}{x^2+y^2}$

$$\therefore \quad u = \begin{cases} \dfrac{x^3 - 3xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$$v = \begin{cases} \dfrac{y^3 - 3x^2 y}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Then

$$u_x(0,0) = \lim_{\Delta x \to 0} \frac{u(0+\Delta x, 0) - u(0,0)}{\Delta x} = 1 \quad (\text{Ex})$$

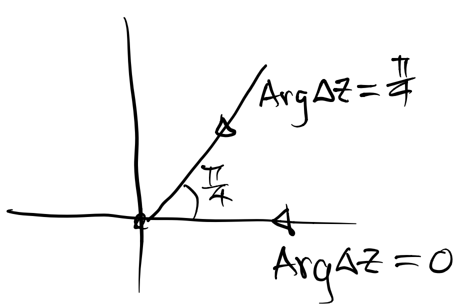
$$\left.\begin{aligned} u_y(0,0) &= \quad \cdots \quad = 0 \\ v_x(0,0) &= \quad \cdots \quad = 0 \\ v_y(0,0) &= \quad \cdots \quad = 1 \end{aligned}\right\}$$

$$\therefore \begin{cases} u_x(0,0) = v_y(0,0) = 1 \\ u_y(0,0) = -v_x(0,0) = 0 \end{cases} \quad \therefore \text{ C-R eqt satisfied at } (0,0).$$

However

$$\frac{f(0+\Delta z) - f(0)}{\Delta z} = \frac{\dfrac{\overline{(\Delta z)}^2}{\Delta z} - 0}{\Delta z}$$

$$= \left(\frac{\overline{(\Delta z)}}{\Delta z} \right)^2 = e^{-i4 \operatorname{Arg} \Delta z}$$



$$= \begin{cases} 1 & \text{for horizontal approach } \left(\text{i.e. } \Delta z = \Delta x + i0, \ \operatorname{Arg} \Delta z = 0. \right) \\ -1 & \text{for diagonal approach } \left(\text{i.e. } \Delta z = \Delta x + i\Delta x \ (\Delta y = \Delta x), \ \operatorname{Arg} \Delta z = \frac{\pi}{4} \right) \end{cases}$$

$$\therefore \lim_{\Delta z \to 0} \frac{f(0+\Delta z) - f(0)}{\Delta z} \text{ doesn't exist.}$$

§23 Sufficient Conditions for Differentiability

<u>Thm</u>: Let $f(z) = u(x,y) + iv(x,y)$ defined throughout some ε-nbd $B_\varepsilon(z_0)$ of $z_0 = x_0 + iy_0$, and

(a) u_x, u_y, v_x, v_y exist <u>everywhere</u> in $B_\varepsilon(z_0)$

(b) u_x, u_y, v_x, v_y are <u>cts</u> at (x_0, y_0) and satisfy $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$ at (x_0, y_0).

Then $f'(z_0)$ exists and $f'(z_0) = (u_x + iv_x)_{(x_0,y_0)}$

Pf: Conditions $\Rightarrow$ u, v differentiable at (x_0, y_0)

Hence together with CR eqt., the <u>optimal exercise</u> $\Rightarrow$ $f'(z_0)$ exists.

& hence the formula. ※

eg: $f(z) = e^x \cos y + i e^x \sin y$ for $z = x+iy$

Then $\begin{cases} u = e^x \cos y \\ v = e^x \sin y \end{cases}$

$\Rightarrow \begin{cases} u_x = e^x \cos y \\ u_y = -e^x \sin y \end{cases}$ $\begin{cases} v_x = e^x \sin y \\ v_y = e^x \cos y \end{cases}$

exist $\forall (x,y) \in \mathbb{R}^2$, and cts, and satisfy

C-R eqt.

$\therefore \Rightarrow$ $f'(z)$ exist and

$$f'(z) = u_x + i v_x = e^x \cos y + i e^x \sin y$$
$$= f(z)$$

$\left(\text{Note } f(z) = e^z = e^{x+iy} \overset{\text{def}}{=} e^x e^{iy} = e^x(\cos y + i \sin y) \right)$

<u>eg 2</u> (same as eg 2 in previous section)

$$f(z) = |z|^2 = x^2 + y^2$$

$$\Rightarrow \begin{cases} u = x^2 + y^2 \\ v = 0 \end{cases}$$

$$\begin{cases} u_x = 2x & v_x = 0 \\ u_y = 2y & v_y = 0 \end{cases}$$ exist $\forall (x,y)$ & cts

$$\begin{cases} u_x(0,0) = v_y(0,0) \\ u_y(0,0) = -v_x(0,0) \end{cases}$$ satisfy C-R eqt at $(0,0)$

& in fact only at $(0,0)$

$\Rightarrow$ $f'(0)$ exists & $f'(0) = u_x(0,0) + i v_x(0,0)$
$= 0$

& $f'(z)$ doesn't exist for $z \neq 0$.

eg3 (Reading Ex!)

§24 Polar coordinates

Thm Let $f(z) = u(r,\theta) + iv(r,\theta)$ be defined in some ε-nbd of a nonzero pt, $z_0 = r_0 e^{i\theta_0}$, and suppose that

(a) $u_r, u_\theta, v_r, v_\theta$ exists everywhere in ε-nbd

(b) $u_r, u_\theta, v_r, v_\theta$ cts at (r_0, θ_0) satisfying

$$\begin{cases} u_r = \frac{1}{r} v_\theta \\ \frac{1}{r} u_\theta = -v_r \end{cases}$$

the Polar form of C-R eqt. at (r_0, θ_0)

Then $f'(z_0)$ exists and

$$f'(z_0) = e^{-i\theta_0}\left(u_r(r_0,\theta_0) + i v_r(r_0,\theta_0)\right).$$

(Pf : Ex : easy change of variables.)

eg: If $f(z) = \frac{1}{z^2} = \frac{1}{r^2 e^{2i\theta}} = \frac{1}{r^2} e^{-2i\theta}$

$$= \frac{1}{r^2}\cos 2\theta - \frac{i}{r^2}\sin 2\theta$$

$$\begin{cases} u_r = -\frac{2}{r^3}\cos 2\theta & v_r = \frac{2}{r^3}\sin 2\theta \\ \frac{1}{r}u_\theta = -\frac{2}{r^3}\sin 2\theta & \frac{1}{r}v_\theta = -\frac{2}{r^3}\cos 2\theta \end{cases}$$

$$\therefore \begin{cases} u_r = \frac{1}{r} v_\theta \\ \frac{1}{r} u_\theta = -v_r \end{cases}$$

Polar form of C-R eqts.

$u_r, u_\theta, v_r, v_\theta$ exist $\forall (r,\theta)$ & cts.

Thm $\Rightarrow$

$f'(z)$ exists &

$$f'(z) = e^{-i\theta}(u_r + i v_r)$$

$$= e^{-i\theta}\left(-\frac{2}{r^3}\cos 2\theta + i\,\frac{2}{r^3}\sin 2\theta\right)$$

$$= -\frac{2}{r^3}\, e^{-i\theta}(\cos 2\theta - i \sin 2\theta)$$

$$= -\frac{2}{r^3}\, e^{-3i\theta} = -\frac{2}{(re^{i\theta})^3} = -\frac{2}{z^3}.$$

eg 2 (Reading Ex!)